Prepared for submission to JCAP

Solar System Gravity

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1 Introduction: Bridging the Gap

Gravity is the force that governs the structure of the universe: it makes the universe expand, it makes the planets orbit the Sun and it makes stars collapse into black holes. Introductory courses on general relativity are commonly taught on undergraduate courses and typically include black hole solutions such as Schwarzchild space-times and cosmological solutions such as the Friedmann-Robertson-Walker (FRW) class of metrics. These solutions are very esoteric: the FRW metric is only important on scales much larger than the solar system and black holes are hard to observe. The aim of this course is to provide a brief introduction to gravity in a familiar environment: the solar system. This is practical general relativity. Rather than looking for esoteric solutions, we will instead solve Einstein's equations in the non-relativistic limit. This is no easy task. The path to observations in theories of gravity is the following:

$$\underbrace{G_{\mu\nu}}_{\text{Field Equations}} \Rightarrow \underbrace{g_{\mu\nu}}_{\text{Metric}} \Rightarrow \underbrace{\Gamma^{\alpha}_{\mu\nu}}_{\text{Geodesic Equations}} \Rightarrow \underbrace{\ddot{x}}_{\text{Motion}}. \tag{1.1}$$

Ultimately, we would like to test general relativity in the solar system and this means that we need alternate theories to provide differing predictions. This means following the above path every time someone comes up with a new theory of gravity. Luckily, we have two sources of aid. The first is a formalism for testing gravity in the solar system developed in the seventies: the parametrised post-Newtonian formalism (PPN). This parametrises the metric in terms of 10 numbers, which completely determine the dynamics of one or many bodies. This means we don't need to follow the whole path but can stop once we have found the metric. The second is the nature of the solar system itself. The motion of day-to-day objects—the Earth, the Moon, the planets—is highly non-relativistic. The solar system is a set of non-relativistic particles moving in a weak gravitational field. But what exactly does this mean? To move non-relativistically means that the speed $v \ll c$ and applying the virial theorem to Newtonian gravity one has

$$v^2 = \Phi_{\rm N}, \tag{1.2}$$

where $\Phi_{\rm N}$ is the Newtonian potential. Now the Earth has an average orbital speed of 30 km/s and so this implies that $\Phi_{\rm N} < 10^{-8}$. Non-relativistic motion implies that one must be moving in a weak gravitational field. In the context of general relativity, $\Phi_{\rm N}$ is the first-order perturbation to the metric and so this tells us that the space-time at the radius of the Earth deviates from Minkowski by one part in 10^8. Since this was found using purely Newtonian physics, any deviations from the Newtonian behaviour must be sub-dominant by a factor of 10^8. This means we don't need to find exact solutions for the metric, we only need to solve for the metric up to post-Newtonian order because our experiments are currently not sensitive enough to measure deviations at the 10^{-16} level. Things are looking up. The first part of this course looks at the solution of Einstein's equations to post-Newtonian order in the solar system. We will then introduce the PPN formalism and apply it to alternate theories of gravity.

Modern alternate theories of gravity include a clever trick to hide modifications locally. They're known as screening mechanisms and they employ non-linear effects to hide modifications of general relativity in the solar system that are active on large scales and determine the cosmological solutions of the theory. PPN doesn't work for these theories because it is a systematic expansion that fails for non-linear theories and so the next part of the course is aimed at introducing these mechanisms and looking at alternate methods of testing them in the solar system. One of the best ways that has emerged recently is the use of non-relativistic stars and the final part of the course gives a brief introduction to the structure and evolution of these stars and shows how changing the theory of gravity can greatly alter their behaviour.

1.1 Reading Material and Conventions

The course is entirely self-contained in these lecture notes but the following extra sources may be useful:

- C. M. Will — *Theory and Experiment in Gravitational Physics*
- E. Poisson & C. M. Will — *Gravity: Newtonian, Post-Newtonian, Relativistic*

- R. Wald — *General Relativity*
- D. Prialnik — *An Introduction to the Theory of Stellar Structure and Evolution*

These notes follow the conventions of Will with one exception: we will not set the value of Newton's constant to unity. In particular: the metric convention is $(-,+,+,+)$ and the spherically symmetric perturbed Minkowski space-time in general relativity is

$$\mathrm{d}s^2 = \left(-1 + 2\frac{G_\mathrm{N}M}{r}\right) + \left(1 + 2\frac{G_\mathrm{N}M}{r}\right)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{1.3}$$

2 Fundamentals of General Relativity

John Wheeler famously said

> "Matter tells space-time how to curve, and curved space-time tells matter how to move."

In this section we will elucidate what this really means in the context of theories of gravitation.

2.1 Einstein's Equations

This course is aimed at studying solar system scale tests of alternative theories of gravity and so we will always start with a Lagrangian description of gravity. Einstein's general relativity is described by the Einstein-Hilbert Lagrangian:

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{R}{16\pi G_\mathrm{N}} + S_\mathrm{m}[g_{\mu\nu}], \tag{2.1}$$

where S_m represents the various particles in the standard model. Varying with respect to the metric $g_{\mu\nu}$ yields Einstein's equations

$$G_{\mu\nu} = 8\pi G_\mathrm{N} T_{\mu\nu}, \tag{2.2}$$

where the energy-momentum tensor

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_\mathrm{m}}{\delta g_{\mu\nu}} \tag{2.3}$$

and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{2.4}$$

is the Einstein tensor with $R_{\mu\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$ being the Ricci tensor and scalar respectively. Defining the trace of the energy-momentum tensor $T \equiv g_{\mu\nu}T^{\mu\nu}$, it is often easier to work with the trace-reversed Einstein equations

$$R_{\mu\nu} = 8\pi G_\mathrm{N}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{2.5}$$

These equations contain some deep physics. The left hand sides contain geometrical tensors that describe the geometry of space-time whereas the right hand side describes the energy content of every day matter. This is the first part of the Wheeler quote above: given any distribution of energy/momentum, the geometry of the space-time is fixed by these equations.

2.2 Gauge Invariance

The Einstein-Hilbert action has a special symmetry. Note that all of the fields depend on x^μ, our coordinates. It turns out that if one redefines the coordinates x^μ in terms of some other coordinates $\tilde{x}^\mu$ so that $\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$, the action (2.1) is invariant. We can use any set of coordinates to describe our space-time and every choice describes the same physics, it is up to us to decide which choice is most convenient for what we are doing. Local symmetries such as this are known as *gauge symmetries* and this specific type of coordinate-invariance is known as *diffeomorphism invariance*. In fact, the word symmetry is a bit of a misnomer, really this is a redundancy in the choice of variables we are using. Theoretically, diffeomorphism invariance is required by special relativity: the fact that gravity is a massless spin-2 particle means that it can have only two helicities, ± 2. Now the metric is a symmetric 4×4 matrix and hence has 10 independent components. Einstein's equations include four constraints and diffeomorphism invariance allows us to fix four more components which leaves us with two propagating degrees of freedom. A specific choice of coordinates is known as *fixing a gauge*. One must fix the gauge before making any physical predictions otherwise what looks like an interesting theoretical prediction may actually be a silly choice of coordinates.

Diffeomorphism invariance also tells us that the energy-momentum tensor is conserved. Recall that under a coordinate transformation the metric transforms as

$$\hat{g}_{\mu\nu}(\tilde{x}^\alpha) = \frac{\partial x^\sigma}{\partial \tilde{x}^\mu}\frac{\partial x^\lambda}{\partial \tilde{x}^\nu}\tilde{g}_{\sigma\lambda}(x^\alpha(\tilde{x}^\rho)). \tag{2.6}$$

Now consider a linearised transformation such that $x^\mu \to x^\mu + \xi_\mu$. The metric transforms as $g_{\mu\nu} \to g_{\mu\nu} + \partial_\mu \xi_{\mu\nu} + \partial_\nu \xi_\mu$, in which case we have

$$\delta S_{\rm m} = \int \mathrm{d}^4 x \frac{\delta S_{\rm m}}{\delta g_{\mu\nu}(x)}\delta g_{\mu\nu}(x) = 2\int \mathrm{d}^4 x \frac{\delta S_{\rm m}}{\delta g_{\mu\nu}(x)}\partial_\mu \xi^\nu = -\int \mathrm{d}^4 x \sqrt{-g}\xi_\nu \nabla_\mu T^{\mu\nu}, \tag{2.7}$$

where we have integrated by parts in the final manipulation. If the matter action is to be invariant under diffeomorphisms we must have energy-momentum conservation:

$$\nabla_\mu T^{\mu\nu} = 0. \tag{2.8}$$

2.3 Motion of Particles in Curved Space-Times

The metric tells us the shape of space-time. If we define $\mathrm{d}s^2$ as the squared line-element connecting two points then one has

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{2.9}$$

In this course, we will be interested in the motion of particles through space-time. Recall that what we call space-time is not really something physical, it is a gauge-choice so our Newtonian view of particles described by time-dependent positions needs to go out the window. In curved space-times, what is important is who is observing; different observers will make different measurements that may not necessarily agree and there is no notion of absolute time.

We would like to describe the motion of some object moving through space-time, which is to be thought of as some trajectory that passes through a series of points x_i^μ. What we need is a continuous parameter λ that describes every point on the trajectory. As an example, consider the two-dimensional space described by coordinates x and y. Now suppose there is an object that moves on a circle of radius R. We can describe this circle using the parametric equations

$$x(t) = R\cos\left(\frac{t}{2\pi}\right) \quad \text{and} \quad y(t) = R\sin\left(\frac{t}{2\pi}\right). \tag{2.10}$$

The parameter $t \in [0, 2\pi]$ tells us where on the circle we are and shifting it by an infinitesimal value moves us continuously along the curve. In a curved space time, we have the coordinates x^μ and we describe the trajectory using four expressions $x^\mu(\lambda)$ in terms of the parameter λ. The trajectory of any particle is known as its *world line* and the proper-time for an observer moving along their world-line is defined as

$$\mathrm{d}\tau = g_{\mu\nu}(x^\mu(\lambda))\,\mathrm{d}x^\mu(\lambda)\,\mathrm{d}x^\nu(\lambda). \tag{2.11}$$

In this course, we will consider only theories of gravity where particles move on geodesics of the metric, and hence satisfy the *geodesic equation*

$$\ddot{x}^\alpha + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{2.12}$$

where a dot denotes a derivative with respect to the parameter λ. Any parameter such that $x^\mu(\lambda)$ satisfies this geodesic equation is called an *affine* parameter. In any coordinate basis, the Christoffel symbols[1] are given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}\right). \tag{2.13}$$

One can see that this equation fully determines the trajectory of a particle but only after the metric and hence the geometry has been specified. This is the second part of the quote above: curved space-time tells matter how to move.

Next, we need to define the notion of velocity. We cannot simply define a particle's velocity as $v^i = \mathrm{d}x^i/\,\mathrm{d}t$ because x and t are a gauge choice and nothing more, they are not physical. The relativistic generalisation of velocity is the 4-velocity

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{2.14}$$

Massless particles such as photons move on null geodesics, which have $g_{\mu\nu}u^\mu u^\nu = 0$, whereas massive particles follow time-like geodesics and one has $g_{\mu\nu}u^\mu u^\nu = -1$.

3 Newtonian and Post-Newtonian Gravity

General relativity is a fully relativistic theory but most situations we can think of involve non-relativistic objects, those which have velocities (relative to us) $v^i \ll c$. What does this mean in general relativity? We've already said that we should not define velocities as $\mathrm{d}x^i/\,\mathrm{d}t$ so how can we make contact with Newtonian physics where time is not a coordinate but is rather a parameter that describes the motion of particles?

To begin with, we want to ask what it means for an object to be non-relativistic and so we will make contact with a simpler theory: special relativity. In special relativity the space-time is fixed to be Minkowski so that

$$g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1) \quad \mathrm{d}s^2 = -\,\mathrm{d}t^2 + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{3.1}$$

[1]Note that I use the word *symbol* because $\Gamma^\alpha_{\mu\nu}$ are not tensors under diffeomorphisms.

Now it turns out that this is a vacuum solution of Einstein's theory of general relativity, i.e. this is the solution when no matter is present so that $T^{\mu\nu} = 0$. Let's look at the motion of particles with respect to this space-time from a general relativity point of view. We have already specified our coordinates and so we have fixed the gauge. Now consider a test-particle moving along its world-line with 4-velocity u^μ. We have

$$u^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right) = \frac{\mathrm{d}t}{\mathrm{d}\tau}\left(1, \frac{\mathrm{d}x^i}{\mathrm{d}t}\right) = \gamma\left(1, \frac{\mathrm{d}x^i}{\mathrm{d}t}\right), \tag{3.2}$$

where we have defined $\gamma = \mathrm{d}t/\mathrm{d}\tau$. Note that since $t = t(\tau)$ we can always invert this relation such that $x^i = x^i(t(\tau))$. Looked at this way, $x^i(t)$ can be seen as the position of the particle at point t as seen by an observer moving with proper-time τ. $v^i = \mathrm{d}x^i/\mathrm{d}t$ is then the rate of change of the object's position as measured by an observer whose proper-time is τ. We will refer to it as the coordinate velocity. Using the condition $g_{\mu\nu}u^\mu u^\nu = -1$ we have

$$\gamma^2(1 - v^i v_i) = 1 \Rightarrow \gamma = \frac{1}{\sqrt{1 - v^2}}, \tag{3.3}$$

where $v^2 = v_i v^i$. γ is none other than the Lorentz factor that is familiar from special relativity. In the context of general relativity it appears as the rate of change of the objects position in the t-direction with respect to the proper time of the observer. In special relativity, being non-relativistic is identical to saying that $v^i \ll c$ and here we can see that the same is true in general relativity provided that we interpret v^i as the coordinate-velocity and not the velocity defined using some notion of absolute time.

Minkowski space has vanishing Christoffel symbols and so the geodesic equation is simply

$$\ddot{x}^\mu = 0. \tag{3.4}$$

We are perfectly at liberty to use proper-time as our affine parameter λ, in which case the $\mu = 0$-component of this tells us that $\mathrm{d}^2t/\mathrm{d}\tau^2 = 0$. The i-component reads

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}^2t}{\mathrm{d}\tau^2} + \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = \gamma^2\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = 0. \tag{3.5}$$

This tells us that test-bodies moving in a flat space-time move with constant coordinate-velocity with respect to observers, which is what we already know from special relativity. In the context of general relativity, we can see that the fact that space-time is not curved means there is no acceleration and objects move on straight lines.

3.1 Newtonian Gravity

We now understand how objects move in flat space-times, but one thing that was missing from our theory of gravity was gravity itself. Note that G_N did not appear at all in our previous analysis. We now want to extend the notion of being non-relativistic to curved space-times and we can do this by recalling what we know about Newtonian gravity. Newtonian gravity is a scalar theory of gravity, it has one gravitational field, the Newtonian potentials Φ_N. This satisfies the Poisson equation

$$\nabla^2\Phi_\mathrm{N} = -4\pi G_\mathrm{N}\rho. \tag{3.6}$$

This equation tells us what the gravitational field of some density distribution looks like but it does not tell us how particles respond to it. For this, we need another equation: Newton's second law

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = \nabla^i \Phi_{\mathrm{N}}. \tag{3.7}$$

Both of these equations should be reproduced in the non-relativistic limit but so far none of them contain the coordinate velocity v^i. Integrating equation (3.7) gives

$$v^2(t) \sim \Phi_{\mathrm{N}}(t), \tag{3.8}$$

where we have integrated with respect to time and are hence evaluating the Newtonian potential along the particle's trajectory. Now the non-relativistic limit corresponds to $v^i \ll c$ (recall $c = 1$ above) and so this is telling us that in order to be non-relativistic, we must have $\Phi_{\mathrm{N}} \sim v^2$. The general solution of (3.6) is

$$\Phi_{\mathrm{N}} = U \quad \text{with} \quad U \equiv G_{\mathrm{N}} \int \mathrm{d}^3 x' \frac{\rho(x')}{|x - x'|} \tag{3.9}$$

and so this tells us that $U \sim v^2$ and hence $\rho \sim v^2$. For a spherically symmetric system, one has

$$U = \frac{GM}{r} \quad M = 4\pi \int r^2 \rho(r) \, \mathrm{d}r, \tag{3.10}$$

which tells us that $v^2 = G_{\mathrm{N}} M/r$. In Newtonian physics, this is the famous relation that tells us how planets move. In general relativity, it is a consistency condition that tells us how different quantities should scale in the non-relativistic limit. We will define the Newtonian limit of any gravity theory as the limit that reproduces equations (3.6) and (3.7) given these scalings for ρ and U. The post-Newtonian limit is the same theory but keeping the next-to-leading-order terms in v/c. From here on we will use the following notation to simplify the discussion. Objects that are of order $\mathcal{O}(1/c^2)$ will be denoted as $\mathcal{O}(1)$. This is to make contact with the literature, which refers to these objects as 1st order in the post-Newtonian (1PN) expansion. $\mathcal{O}(1.5)$ objects scale like $1/c^3$ and $\mathcal{O}(2)$ like $1/c^4$. In practice, solar system objects have $v^2/c^2 \sim 10^{-5}$–10^{-10} and so post-Newtonian effects are sub-dominant to the Newtonian behaviour by at least three orders-of-magnitude, generally more. Despite this, because we observe over long time-scales (of order many orbits) post-Newtonian effects can be non-negligible and, indeed, must be accounted for in certain situations such as calculating the orbit of mercury.

Our next job is to work out how to reproduce the above equations in a general relativity context. We know that Minkowski space is a vacuum solution of Einstein's equations so now we need to add some non-relativistic object described by $\rho(r)$ and see what happens. Recall that the energy-momentum tensor is given by

$$T^{\mu\nu} = \rho(1 + \Pi + \frac{P}{\rho}) u^\mu u^\nu + P g^{\mu\nu}, \tag{3.11}$$

where P is the pressure and Π is the internal energy per unit mass[2]. Since we have decided

[2]This is neglected in cosmology but it can be important in certain theories of gravity and so it is retained by those studying solar system physics.

that $\rho \sim \mathcal{O}(1)$ we need to impose that $P\,,\Pi \sim \mathcal{O}(2)$[3]. With these definitions, one simply has

$$T^{\mu\nu} = \rho u^\mu u^\nu \tag{3.12}$$

at Newtonian order.

Next, we need to decide what happens to the metric. We write the perturbed metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu} \sim \mathcal{O}(1)$. In theory, we should solve for every component of $h_{\mu\nu}$ but we know that the theory is gauge invariant and so we can make a suitable choice of coordinates to make this easier. We will choose a gauge where $h_{0i} = 0$ and h_{ij} is diagonal. In this case, we have two metric potentials left to solve for and we will define them as follows: $h_{00} = 2\chi_1$, $h_{ij} = 2\Psi\delta_{ij}$. The total metric at 1PN is then[4]

$$\mathrm{d}s^2 = (-1 + 2\chi_1)\,\mathrm{d}t^2 + (1 + 2\Psi)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{3.13}$$

Now if we want to make contact with the equations of Newtonian gravity we have to somehow relate the relativistic formalism to the coordinate velocity seen by an observer whose proper-time is τ, which means we need to relate u^μ to v^i. Since ρ is already $\mathcal{O}(1)$ we only need u^μ to $\mathcal{O}(0)$ but we have already done this above where we found $u^\mu = \gamma(1, v^i)$ and $\gamma = (1 - v^2)^{-1/2} = 1$ to $\mathcal{O}(1)$ and so to $\mathcal{O}(0)$ one simply has $u^\mu = (1, 0)$. This should hardly come as a surprise. The Newtonian limit is both the weak field limit of gravity and the non-relativistic limit of special relativity and so v/c corrections should be absent. With this in mind, we simply have $T^{00} = \rho$, $T^{0i} = T^{ij} = 0$.

We first want to solve Einstein's equations to $\mathcal{O}(1)$ and this is best done using the trace-reversed form of Einstein's equations (2.5). To $\mathcal{O}(1)$, one has $T_{00} = g_{00}^2 T^{00} = \rho$ and $T = g_{00}T^{00} = -\rho$. The 00-component gives $R_{00} = -\nabla^2\chi_1$ and so one has

$$\nabla^2\chi_1 = -4\pi G_{\mathrm{N}}\rho. \tag{3.14}$$

This is precisely the Poisson equation of Newtonian gravity. The ij-component of the trace-reversed equations give

$$\nabla^2\Psi = -4\pi G_{\mathrm{N}}\rho \tag{3.15}$$

and so the solution at 1PN is $\chi_1 = \Psi = U = \Phi_{\mathrm{N}}$.

This takes care of the field equation for Φ_{N} but we also need to recover Newton's law. This is done using the geodesic equation. With the perturbed space-time, one still has most of the Christoffel symbols vanishing but not all of them. The non-vanishing ones must be $\mathcal{O}(1)$ or higher and so only terms of the form $\Gamma^\mu_{00}\dot{x}^0\dot{x}^0$ survive. The reason for this is that $\dot{x}^i = \gamma v^i = v^i \sim \mathcal{O}(0.5)$[5] and hence terms like $x^i\Gamma^\mu_{i\nu}$ must be $\mathcal{O}(1.5)$ or higher. One finds that $\Gamma^0_{00} \sim \mathcal{O}(1.5)$ and so again we have $\ddot{x}^0 = 0$ just as in special relativity but $\Gamma^i_{00} = -\nabla^i\chi_1$ and so the i-component of the geodesic equation is

$$\gamma^2\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} - \nabla^i\chi_1 = 0 \Rightarrow \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = \nabla^i\Phi_{\mathrm{N}}. \tag{3.16}$$

This is precisely newton's second law.

[3]One may wonder why not $\mathcal{O}(1.5)$. In the case of pressure, one would like to retain the Euler-equations from hydrodynamics, which read $\rho\,\mathrm{d}v^i/\,\mathrm{d}t = -\rho\nabla U - \nabla P$. The left hand side is $\mathcal{O}(2)$ as is the first term on the right provided that $\rho \sim \mathcal{O}(1)$ and hence $P \sim \mathcal{O}(2)$ for consistency. In the case of Π, this is motivated by the numerical value of Π for solar system objects.

[4]Note that the 00-component has a subscript 1 whereas the ij-component does not. The reason for this will become apparent later. In short, it is because the standard gauge for performing post-Newtonian calculations requires g_{00} to $\mathcal{O}(2)$ but g_{ij} to $\mathcal{O}(1)$ only, hence the need to distinguish metric potentials in g_{00} but not g_{ij}.

[5]Recall $v^2 \sim \mathcal{O}(1)$ so that $v^i \sim \mathcal{O}(0.5)$.

3.1.1 Light Bending

So far, we have just come up with a very complicated theory that predicts Newton's laws in some limit but a good theory is falsifiable: it must make new predictions. We have not yet used the second metric potential Ψ. This is a new prediction of general relativity so let's see what its consequences are. First note that it does not appear in the non-relativistic equations of motion and so we must study the motion of relativistic particles: light. Light moves on null geodesics and so $\mathrm{d}s^2 = 0$. This means we cannot use the proper-time as a parameter but must instead use some affine parameter λ. We can always use the coordinate time t to make measurements because $t = t(\lambda)$ along the photon's world-line and so we can define $v^\alpha = \mathrm{d}x^\alpha / \mathrm{d}t = (1, \frac{\mathrm{d}x^\alpha}{\mathrm{d}t})$. So why do we need to use light to probe the metric potential Ψ? The answer can be seen by looking at the spatial-component of the geodesic equation:

$$\frac{\mathrm{d}v^i}{\mathrm{d}t} + \Gamma^i_{\mu\nu} v^\mu v^\nu + \cdots = 0, \tag{3.17}$$

where $\cdots$ stand for terms that appear when we change from the affine parameter λ to $t(\lambda)$. We know that $\Gamma^i_{0j} v^k \sim \mathcal{O}(1.5)$ and $\Gamma^i_{jk} v^j v^k \sim \mathcal{O}(2)$ for non-relativistic particles and so the only contribution is from Γ^i_{00}, which depends on χ_1 only. This was a result of the fact that $v^i \sim \mathcal{O}0.5)$ for non-relativistic particles. Massless particles, on the other hand, move at the speed of light and so we cannot treat v^i as a small parameter. For this reason, the other components of the Christoffel symbols, which do depend on Ψ, contribute to their motion. In general relativity we have $\Psi = U$. We won't show this here but if one calculates the motion of photons to Newtonian order the effect of the curved space-time is to bend their trajectory by an angle:

$$\theta = \frac{4G_\mathrm{N}M}{b}, \tag{3.18}$$

where b is the impact parameter. This is shown in figure 1. This is our first prediction of general relativity that could not be derived from Newtonian physics. It is a firm prediction in that it only cares about parameters we can measure: G_N, M, and b. Given the value of G_N from non-relativistic physics, this effect is either there or not. Measuring the angle by which light is bent by the Sun therefore constitutes a test of general relativity and not Newtonian physics. This course is about alternative theories of gravity. Suppose that we were to find an alternate theory of gravity where $\chi_1 = 2U$ but $\Psi_1 = 2\gamma_\mathrm{PPN}U$, where γ_PPN is a constant known as the *Eddington light-bending parameter*[6]. One could repeat the calculation of the motion of photons in this modified geometry to find

$$\theta = \left(\frac{1+\gamma_\mathrm{PPN}}{2}\right)\frac{4G_\mathrm{N}M}{b}. \tag{3.19}$$

One can see that when the theory of gravity is general relativity we have $\gamma = 1$ and so an experimental measurement of $\gamma \neq 1$ would be a signal of modified gravity. In practice, this parameter is constrained to be smaller than $\sim 10^{-5}$ and so light-bending in alternate theories of gravity must be very close to general relativity. We will return to this later on.

3.2 Post-Newtonian Gravity

We have seen that at 1PN there is only one new parameter, γ_PPN, that characterises deviations from general relativity in the solar system. This is not an accident. Einstein's

[6]The PPN stands for "parametrised post-Newtonian". We will see what this means in the next section.

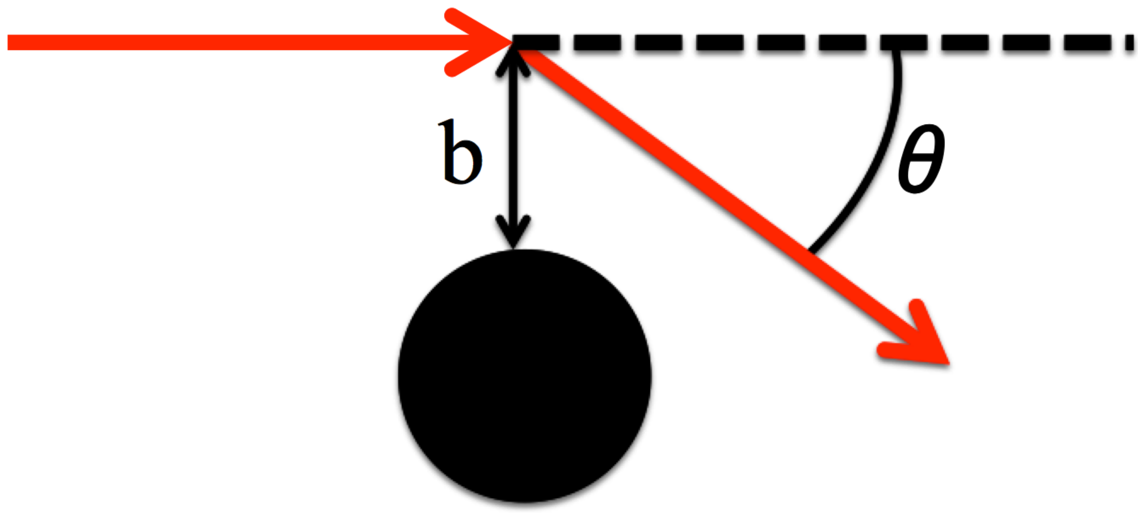

Figure 1. Light bending by a spherical body.

equations tell us that the metric around some matter distribution should be sourced by its energy-momentum distribution and hence we expect metric potentials to depend on quantities such as the density, pressure and internal velocity. Furthermore, since $g_{\mu\nu}$ is a tensor under general coordinate transformations we know that h_{00} should transform as a scalar, h_{0i} should transform as a vector and h_{ij} should transform as a tensor. The only 1PN quantities we can define that have these properties are U, defined in (3.9) and

$$U_{ij} \equiv G_{\rm N} \int \mathrm{d}^3 x' \frac{\rho(x')(x-x')_i(x-x')_j}{|x-x'|^3}. \tag{3.20}$$

U_{ij} does not appear as a consequence of our gauge choice i.e. we chose a gauge where h_{ij} was diagonal.

It turns out that at post-Newtonian (2PN) order the following metric potentials satisfy all of the required properties:

$$\begin{aligned}
\Phi_1 &\equiv G \int \mathrm{d}^3\vec{x}' \frac{\rho(\vec{x}')v^2(\vec{x}')}{|\vec{x}-\vec{x}'|}, \quad \Phi_2 \equiv G \int \mathrm{d}^3\vec{x}' \frac{\rho(\vec{x}')U(\vec{x}')}{|\vec{x}-\vec{x}'|}, \\
\Phi_3 &\equiv G \int \mathrm{d}^3\vec{x}' \frac{\rho(\vec{x}')\Pi(\vec{x}')}{|\vec{x}-\vec{x}'|}, \quad \Phi_4 \equiv G \int \mathrm{d}^3\vec{x}' \frac{p(\vec{x}')}{|\vec{x}-\vec{x}'|}, \\
V_i &\equiv G \int \mathrm{d}^3\vec{x}' \frac{v_i(\vec{x}')\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}, \quad W_i \equiv G \int \mathrm{d}^3\vec{x}' \frac{\rho(\vec{x}')\vec{v}\cdot(\vec{x}-\vec{x}')(x-x')_i}{|\vec{x}-\vec{x}'|^3}, \\
\Phi_W &\equiv \int \mathrm{d}^3x'\,\mathrm{d}^3x'' \frac{\rho(x')\rho(x'')}{|\vec{x}-\vec{x}'|^3} \cdot \left(\frac{\vec{x}'-\vec{x}''}{|\vec{x}-\vec{x}''|} - \frac{\vec{x}-\vec{x}''}{|\vec{x}'-\vec{x}''|} \right) \quad \text{and} \\
\mathcal{A} &\equiv \int \mathrm{d}^3x' \frac{\rho(x')\left[\vec{v}(x')\cdot(\vec{x}-\vec{x}')\right]^2}{|\vec{x}-\vec{x}'|^3}.
\end{aligned} \tag{3.21}$$

Note here that v^i is the internal velocity field relative to the observer and one should think of it as the Eulerian velocity. These satisfy

$$\begin{array}{ll} \nabla^2 V_i = -4\pi G_{\rm N} v_i & \nabla^2 \Phi_1 = -4\pi G_{\rm N}\rho v^2 \\ \nabla^2 \Phi_2 = -4\pi G_{\rm N}\rho U & \nabla^2 \Phi_3 = -4\pi G_{\rm N}\rho \Pi \\ \nabla^2 \Phi_4 = -4\pi G_{\rm N} P & \partial_i U \partial^i U = \nabla^2(\frac{U^2}{2} - \Phi_2), \\ \partial_0\partial_i U = -\frac{1}{2}\nabla^2(V_i - W_i) & \end{array} \tag{3.22}$$

Let's calculate the post-Newtonian metric for general relativity and see which one of these metric appear. First, we put the gauge choice above on a firmer footing and impose the conditions

$$\partial_\mu h^\mu_0 - \frac{1}{2}\partial_0 h^\mu_\mu = -\frac{1}{2}\partial_0 h_{00} \quad \text{and} \tag{3.23}$$

$$\partial_\mu h^\mu_i - \frac{1}{2}\partial_i h^\mu_\mu = 0. \tag{3.24}$$

When imposed, these conditions fully specify the metric as follows:

$$\begin{array}{ll} g_{00} = -1 + 2\chi_1 + 2\chi_2, & g^{00} = -1 - 2\chi_1 - 2\chi_2 - 4\chi_1^2 \\ g_{0i} = B_i, & g^{0i} = -B^i \\ g_{ij} = 1 + 2\Psi\delta_{ij}, & g^{ij} = \left(1 - 2\Psi + 4\Psi_1^2\right)\delta^{ij}, \end{array}$$

and imposes the conditions

$$\partial_k B^k = 3\partial_0 \Psi \tag{3.25}$$

$$\partial_0 B_k = \partial_k \chi_2. \tag{3.26}$$

Note that χ_1 and Φ_1 are Newtonian and we have calculated them already above. One can see that in our chosen gauge we have $g^{00} \sim \mathcal{O}(2)$, $g^{0i} \sim \mathcal{O}(1.5)$ and $g^{ij} \sim \mathcal{O}(2)$ and so we only need $T^{\mu\nu}$ to this order.

Next, we need the energy-momentum tensor to $\mathcal{O}(2)$. First, we repeat the calculation of γ up to $\mathcal{O}(1)$ using the condition $g_{\mu\nu}u^\mu u^\nu = -1$ with $u^\mu = \gamma(1, v^i)$ to find

$$(-1 + 2\chi_1)\gamma^2 + (1 + 2\Psi)\gamma^2 v^2 = -1 \rightarrow \gamma = \frac{1}{\sqrt{1 - 2\chi_1 - v^2}} + \mathcal{O}(2) = 1 + \chi_1 + \frac{v^2}{2}. \tag{3.27}$$

The presence of a curved space-time alters the Lorentz factor; the factor of χ_1 is the source of gravitational redshift. Using equation (3.11) we then have

$$\begin{aligned} T^{00} &= \rho\left[1 + \Pi + v^2 + 2U\right] \\ T^{0i} &= \rho v^i \quad \text{and} \\ T^{ij} &= \rho v^i v^j + P\delta^{ij}, \end{aligned} \tag{3.28}$$

where we have replaced $\chi_1 = \Psi = 2U$. This is the first point in this calculation that we have specified to the case of general relativity.

Exercise:

Show that:

$$T_{00} = \rho\left[1 + \Pi + v^2 - 2U\right] \tag{3.29}$$
$$T_{0i} = -\rho v_i v_j \tag{3.30}$$
$$T_{ij} = \rho v^i v^j + P\delta^{ij} \quad \text{and} \tag{3.31}$$
$$T = -\rho\left[1 + \Pi + v^2\right] + \rho v^2 + 3P \quad \text{where} \quad T = g_{\mu\nu}T^{\mu\nu}. \tag{3.32}$$

Again, we want to use the trace-reversed Einstein equations. We'll start with the $0i$-component, which gives

$$\nabla^2 B_i + \partial_i\partial_0 U = 16\pi G_{\rm N}\rho v_i. \tag{3.33}$$

Using the identity (3.22), this becomes

$$\nabla^2 B_i = -\nabla^2\left(\frac{7}{2}V_i + \frac{1}{2}W_i\right) \Rightarrow B_i = -\frac{7}{2}V_i - \frac{1}{2}W_i. \tag{3.34}$$

The 00-component gives

$$\nabla^2(\chi_2 + U^2 - 4\Phi_2) = \nabla^2\left(2\Phi_1 - 2\Phi_2 + \Phi_3 + 3\Phi_4\right) \tag{3.35}$$

and so one finds

$$\chi_2 = -U^2 + 2\Phi_1 + 2\Phi_2 + \Phi_3 + 3\Phi_4. \tag{3.36}$$

Exercise:

If one expands $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ show that:

$$R_{00} = -\frac{1}{2}\nabla^2(h_{00} + 2U^2 - 8\Phi_2) + \mathcal{O}(3) \tag{3.37}$$
$$R_{0i} = \nabla^2 h_{0i} + \partial_0\partial_i U + \mathcal{O}(2.5) \tag{3.38}$$
$$R_{ij} = \nabla^2 h_{ij} + \mathcal{O}(2) \tag{3.39}$$
$$\tag{3.40}$$

You will need to use the $\mathcal{O}(1)$ solution $h_{00} = h_{ij} = 2U$ and several of the identities given in (3.22) and (3.25) and (3.26).

Putting it all together, the post-Newtonian metric is

$$\begin{aligned} g_{00} &= -1 + 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 3\Phi_4 \\ g_{0i} &= -\frac{7}{2}V_i - \frac{1}{2}W_i \\ g_{ij} &= (1 + 2U)\delta_{ij}. \end{aligned} \tag{3.41}$$

This is the general relativity prediction for the geometry of space-time up to $\mathcal{O}(1/c^4)$ outside an isolated object. What we could do now is go ahead and try to calculate things like the corrections to the properties of orbits or the bending of light, but, as we remarked in the introduction, this is very cumbersome and there is a standard procedure for comparing experimental measurements with theory once the metric has been found. Furthermore, the specific form of the post-Newtonian metric (3.41) is not very enlightening; there are no specific terms that correspond to individual new effects. For example, we all know that general relativity causes the perihelion of Mercury to precess but the rate of precession is not linked to the presence of one term; several different terms contribute. This is made even more complicated by the fact that post-Newtonian corrections to the two-body problem are of a similar magnitude to the corrections coming from Newtonian sources such as the finite size of the Sun and perturbations from other celestial objects such as Venus and the Earth. For this reason, we turn our attention to the generalised framework. This will allow us to gain a better insight into the general relativity post-Newtonian metric; the terms that could be present but are not tell us a lot about the structure of gravity.

3.3 The Parametrised Post-Newtonian Framework

We can see that not all of the metric potentials that could be present in the metric actually appear in general relativity but there are many alternate theories where they might. Calculating the 2PN metric in general relativity was straight-forward but any quick glance at a text book on post-Newtonian gravity reveals that translating this into the motion of celestial objects is a long and cumbersome process. Rather than have to solve the two-body problem at 2PN for every theory of gravity, theorists in the 70's and 80's developed a parametrised post-Newtonian framework (PPN) for testing gravity in the solar system. The PPN metric in the gauge (3.23) and (3.24), which are known as quasi-Cartesian coordinates, is

$$\begin{aligned}\tilde{g}_{00} =& -1+2U-2\beta U^2-2\xi\Phi_W+(2\gamma_{\rm PPN}+2+\alpha_3+\zeta_1-2\xi)\Phi_1+2(3\gamma_{\rm PPN}-2\beta+1+\zeta_2+\xi)\Phi_2\\ &+2(1+\zeta_3)\Phi_3+2(3\gamma_{\rm PPN}+3\zeta_4-2\xi)\Phi_4-(\zeta_1-2\xi)\mathcal{A}-(\alpha_1-\alpha_2-\alpha_3)w^2U-\alpha_2 w^i w^j U_{ij}\\ &+(2\alpha_3-\alpha_1)w^iV_i, \end{aligned} \tag{3.42}$$

$$\begin{aligned}\tilde{g}_{0i} =& -\frac{1}{2}(4\gamma_{\rm PPN}+3+\alpha_1-\alpha_2+\zeta_1-2\xi)V_i-\frac{1}{2}(1+\alpha_2-\zeta_1+2\xi)W_i-\frac{1}{2}(\alpha_1-2\alpha_2)w^iU\\ &-\alpha_2 w^j U_{ij}, \end{aligned} \tag{3.43}$$

$$\tilde{g}_{ij} = (1+2\gamma_{\rm PPN}U)\delta_{ij}. \tag{3.44}$$

This needs some explanation. First, note that all of the metric potentials (3.21) that we said could potentially appear in the metric are present. This is because we want to encompass as many theories of gravity as possible. The parameter $\gamma_{\rm PPN}$ was discussed above and parametrises deviations from general relativity in the Newtonian limit[7]. In particular, it determines the angle through which light is bent by the Sun. The other nine parameters β, ξ, α_i and ζ_i parametrise deviations from general relativity. In particular, the case $\gamma_{\rm PPN}=\beta=1$, $\xi=\alpha_i=\zeta_i=0$ corresponds to the general relativity prediction we calculated above. Finally, there is the quantity w^i. This is the velocity of the solar system relative to the mean rest frame of the universe. This appears because there may be preferred frame effects in theories of gravity that include aethers or fixed external fields. These are parametrised by the α_i

[7] It also appears in many post-Newtonian expressions, for example, the expression for the perihelion advance of Mercury.

parameters, which are typically non-zero due to the effects of cosmological scalars and vectors. The PPN metric is written in this form rather than placing an arbitrary parameter in front of each metric potential so that each parameter carries a physical interpretation. I will outline this briefly below:

- γ_{PPN} — The amount of spacial curvature produced per unit rest mass. This alters the motion of light relative to non-relativistic particles.
- β — The amount of non-linearity in the superposition law for gravity. This affects the motion of binary objects, for example, it causes a periodic shift in the perihelion of Mercury.
- ξ — The Whitehead parameter. General relativity requires one to solve for the metric before solving for the motion of particles. Alfred North Whitehead saw this as acausal and introduced a new theory in 1922 where the physical metric seen by particles only depends on quantities evaluated along their past light-cone. The theory makes identical predictions to general relativity except that $\xi = 1$ and passes all of the classical tests including light bending and the perihelion shift of mercury. It remained a viable competitor to general relativity until 1971 when it was pointed out that when $\xi \neq 0$ there are anisotropies in the local value of G_{N} in three-body systems. This results in large unobserved tides on Earth due to the motion of the solar system through the Milky Way. ξ is zero in the majority of alternate theories of gravity but is non-zero in *quasi-linear* theories.
- α_1 and α_2 — These describe preferred frame effects. They are typically zero unless the theory violates Lorentz invariance or contains some sort of aether. Theories with non-zero cosmological vectors typically act as an aether.
- ζ_i — These are the so-called *conservation law parameters*. When they are non-zero, the theory lacks the conservation laws that usually arise due to translational invariance such as energy and momentum conservation.
- α_3 — This is both a preferred frame and conservation law parameter.

Not only do these parameters tell us about deviations from general relativity but they also tell us about conservation laws. Recall that the Poincare group (Lorentz group + translations) group has 10 conserved currents P^μ, the energy and momentum and $J^{\mu\nu}$, the angular momentum. It turns out that these are only conserved in a curved space if certain combinations of the parameters are zero. There are three classes of theories

1. Conservative theories: These have $\alpha_i = \zeta_i = 0$ and conserve both P^μ and $J^{\mu\nu}$.
2. Semi-Conservative theories: These have $\zeta_i = \alpha_3 = 0$ and either α_1 or α_2 non-zero. In this case P^μ is conserved but $J^{\mu\nu}$ is not.
3. Non-Conservative theories: These have one of ζ_i or α_3 non-zero and do not conserve any quantities.

There is a theorem that any theory that can be derived from a diffeomorphism-invariant Lagrangian is at least semi-conservative. We end this section by presenting the current bounds on the PPN parameters in table 3.3.

Parameter	Constraint	Experiment
$\gamma_{\rm PPN}-1$	2.5×10^{-5}	Light bending by the Sun measured by the Cassini probe
$\beta-1$	3×10^{-3}	Perihelion shift of Mercury
ξ	10^{-3}	Gravimetric data about the Earth's tides
α_1	10^{-4}	Orbit polarisation measured using Lunar Laser Ranging
α_2	4×10^{-7}	Spin precession of the Sun's axis with respect to its ecliptic
α_3	4×10^{-20}	Pulsar spin-down statistics
ζ_1	0.02	Combined PPN bounds
ζ_2	4×10^{-5}	Binary pulsar acceleration
ζ_3	10^{-8}	Newton's third law measured using the acceleration of the Moon
ζ_4	0.4	Difference in active and passive mass between bromine and fluorine.

3.3.1 The PPN Metric for Scalar-Tensor Theories

As an example of how to apply the PPN formalism, let's look at one of these alternate theories I keep talking about: scalar-tensor theories. These theories include a new scalar field ϕ and are described by the action

$$S=\int\sqrt{-g}\frac{1}{16\pi G}\left[\frac{R}{2}-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\right]+S_{\rm m}[\tilde{g}_{\mu\nu}]. \tag{3.45}$$

The metric that couples to matter is not the *Einstein frame metric* $g_{\mu\nu}$ but instead it is the *Jordan frame metric*

$$\tilde{g}_{\mu\nu}=A^2(\phi)g_{\mu\nu}. \tag{3.46}$$

One very common choice of coupling function is

$$A(\phi)=e^{\alpha\phi}, \tag{3.47}$$

where α is a constant. The equations for the tensor $g_{\mu\nu}$ are simply the Einstein equations sourced by both matter and the scalar

$$G_{\mu\nu}=8\pi G\left(T_{\mu\nu}+T_{\phi,\,\mu\nu}\right) \tag{3.48}$$

where

$$T_{\phi,\,\mu\nu}=\frac{1}{8\pi G}\left(\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi g_{\mu\nu}\right). \tag{3.49}$$

Note that I have used $G\equiv(8\pi {M_{\rm pl}}^2)^{-1}$ here and not $G_{\rm N}$. It is often the case in alternate theories of gravity that what you call G in the action is not the same thing as $G_{\rm N}$, the locally measured value of Newton's constant, and so it is best to distinguish between the two. One also needs the scalar field's equation of motion, which is

$$\Box\phi=-8\pi GT\frac{{\rm d}\ln A}{{\rm d}\phi}, \tag{3.50}$$

where $\Box=g^{\mu\nu}\nabla_\mu\nabla_\nu$. Since it is the Jordan frame metric that couples to matter we need to compute this in the same manner that we did before. The added complication is the scalar. We will make the ansatz that $\phi=\phi_0+\phi_1+\phi_2$ where ϕ_0 is the cosmological value of the field, $\phi_1\sim\mathcal{O}(1)$ is the Newtonian field (1PN) and $\phi_2\sim\mathcal{O}(2)$ is the post-Newtonian field (2PN).

We will set $\phi_0 = 0$ so that $A(\phi_0) = 1$[8]. Note then that the Jordan frame metric is, to the appropriate order,

$$\begin{aligned}\tilde{g}_{00} &= \left(-1 + 2[\chi_1 - \alpha\phi_1] + \left[2\chi_2 + 2\alpha\phi_2 + \alpha^2\phi_1^2\right]\right)\\ \tilde{g}_{0i} &= B_i\\ \tilde{g}_{ij} &= (1 + 2\Psi + 2\alpha\phi_1)\delta_{ij},\end{aligned} \tag{3.51}$$

where the metric potentials have been defined in the Einstein frame according to (3.25). In particular, the $\tilde{g}_{00}$ component at $\mathcal{O}(1)$ is $\chi_1 - \alpha\phi_1$ and so we have an effective value of $\tilde{\chi}_1 = \chi_1 - \alpha\phi_1$. This means that in the Jordan frame, the energy-momentum tensor is given by (3.28) with $\chi_1 \to \tilde{\chi}_1$. The only complication here is that the energy-momentum tensors in the different frames are related via

$$T^{\mu\nu} = A^6(\phi)\tilde{T}^{\mu\nu}. \tag{3.52}$$

The physical quantities such as the coordinate velocity, the density and pressure etc. should be defined in this frame since it is the frame in which gravity is minimally coupled and one has $\nabla_\mu\tilde{T}^{\mu\nu} = 0$. $\nabla_\mu T^{\mu\nu} \neq 0$ in the Einstein frame since the scalar couples to matter and instead we have $\nabla_\mu(T^{\mu\nu} + T_\phi^{\mu\nu}) = 0$. It turns out that at the post-Newtonian level this, theory is indistinguishable from general relativity and that the only change is at the Newtonian level. For this reason, I will only calculate the Jordan frame metric to $\mathcal{O}(1)$. To this order, $\tilde{T}^{\mu\nu} = T^{\mu\nu}$ and so we can use all of the formulae given in the previous section provided that we work to $\mathcal{O}(1)$ only. Also, we do not need to worry about $T_{\phi\,,\mu\nu}$. The reason is the following: expanding it out around the cosmological field value ϕ_0 to $\mathcal{O}(1)$ one finds

$$\begin{aligned}T_\phi^{00} &= 0\\ T_\phi^{0i} &= -\dot{\phi}_0\\ T_\phi^{ij} &= 0.\end{aligned} \tag{3.53}$$

The only non-zero term is in T^{ij} and this is multiplied by $\dot{\phi}_0$. Since ϕ_0 is a cosmological scalar we expect $\dot{\phi}_0 \sim H_0\phi_0 \ll \partial_i h_{\mu\nu}$ and so one can ignore the scalar's contribution to the energy-momentum tensor. This is an important feature of alternate theories of gravity, the local space-time curvature should be sourced by the matter and not the scalar. With this simplification, at $\mathcal{O}(1)$ the Einstein equations are identical to general relativity and so one has

$$\chi_1 = \Psi = U, \tag{3.54}$$

with the caveat that U is defined using G and not $G_{\rm N}$. We will return to this later. Now let's solve for the scalar. To $\mathcal{O}(1)$ we have $T = -\rho$ and so equation (3.50) is

$$\nabla^2\phi_1 = 8\alpha\pi G\rho \Rightarrow \phi_1 = -2\alpha U. \tag{3.55}$$

This is all we need to solve for the $\mathcal{O}(1)$ metric. Putting the solutions for χ_1, Ψ and ϕ_1 into the Jordan frame metric (3.51) we have

$$\begin{aligned}\tilde{g}_{00} &= -1 + 2U(1 + 2\alpha^2)\\ \tilde{g}_{0i} &= 0\end{aligned}$$

[8]We can always rescale the Jordan frame coordinates at zeroth-order so that this is the case.

$$\tilde{g}_{ij} = \left[1 + \left(1 - 2\alpha^2\right) U\right] \delta_{ij}. \tag{3.56}$$

This is not in the PPN form because the coefficient of U in $\tilde{g}_{00} \neq 1$ (see (3.42)). This is because what we called G in the action is not the same as G_N. Since U contains a factor of G we can define $G_\mathrm{N} \equiv (1 + 2\alpha^2)G$, in which case we have

$$\begin{aligned}
\tilde{g}_{00} &= -1 + 2U \\
\tilde{g}_{0i} &= 0 \\
\tilde{g}_{ij} &= \left[1 + 2U\left(\frac{1 - 2\alpha^2}{1 + 2\alpha^2}\right)\right] \delta_{ij},
\end{aligned} \tag{3.57}$$

where U is now defined using G_N. Comparing with (3.44), we see that this is now in PPN form with

$$|\gamma_\mathrm{PPN} - 1| = \frac{4\alpha^2}{1 + 2\alpha^2}. \tag{3.58}$$

Now comes the power of the PPN formalism: Rather than deriving the effects of $\gamma \neq 1$ on the motion of light and particles we know that the strongest constraint on its value in a metric of the PPN form comes from the Cassini probe, which constrains this quantity to be smaller than 10^{-5}. This imposes the constraint

$$4\alpha^2 \lesssim 10^{-5}. \tag{3.59}$$

Scalar-tensor theories are highly constrained.

Exercise:

More general scalar-tensor theories can be parametrised as follows:

$$\ln A(\phi) = \alpha_0(\phi - \phi_0) + \frac{1}{2}\beta_0(\phi - \phi_0)^2 + \ldots. \tag{3.60}$$

Using this parametrisation, calculate the Jordan frame metric to 2PN order and show that

$$\gamma_\mathrm{PPN} = \frac{1 - 2\alpha_0^2}{1 + 2\alpha_0^2}, \quad \beta = \frac{\alpha_0^2\beta_0}{2(1 + 2\alpha^2)^2}, \quad \alpha_i = \zeta_i = \xi = 0. \tag{3.61}$$

You will need to make sure you scale G by the appropriate factors of α_0 when you convert to G_N.

4 Screening Mechanisms

Many people who study alternate theories of gravity are interested in the cosmological constant problem and looking for accelerating solutions but there is no way that a theory like the one that we studied above can possibly have any major effect on the expansion of the universe if the only new parameter α is five orders-of-magnitude smaller than unity. The problem with many alternate theories of gravity is that solar system tests are so strong that they render the theory irrelevant on all scales. What would be nice if there was some sort of mechanism where the scalar's effects are negligible in the solar system but important for cosmology. To see if this is possible, let's look at what went wrong above. Written in the Einstein frame, the theory looked like Einstein's equations plus an extra equation for the scalar. When written in the Jordan frame, we found that the metric potential $\tilde{\chi}_1$, which governs non-relativistic geodesics, was related to the Einstein frame potentials via

$$\tilde{\chi}_1 = \chi_1 - 2\alpha\phi_1. \tag{4.1}$$

Now χ_1 satisfied the Poisson equation of general relativity

$$\nabla^2\chi_1 = -4\pi G\rho \Rightarrow \chi_1 = 2U \tag{4.2}$$

but so did the scalar up to a factor of -2α:

$$\nabla^2\phi_1 = 8\pi G\alpha\rho \Rightarrow \phi_1 = -2\alpha\chi_1. \tag{4.3}$$

When put together, this means

$$\tilde{\chi}_1 = (1+2\alpha^2)\chi_1^{\mathrm{GR}}. \tag{4.4}$$

This means that observers making measurements with respect to the Einstein frame metric see a value of $G_{\mathrm{N}} = (1+2\alpha^2)G$. Screening mechanisms attempt to hide modifications of general relativity by changing the Poisson equation for ϕ such that the solution is very different from U. When written in the Einstein frame, the theory looks like one with a gravitational- and an additional *fifth*-force

$$\vec{F}_{\mathrm{N}} = -\nabla U, \quad \vec{F}_5 = -\alpha\nabla\phi, \tag{4.5}$$

and so the solution of the new Poisson equation should satisfy $\alpha|\nabla\phi|/|\nabla U| \ll 1$ in the Einstein frame. There are two very different approaches to this. One is to somehow kill of the source for the Poisson equation so that no scalar gradients are generated by massive objects, and the other is to change the derivative interactions on the left hand side so that the solution is very different from U. In this lecture we will examine both of these. From here on I will change notation from ϕ_1 to ϕ. The reason for this is that screening mechanisms tend to be non-linear in the field equations and it does not make sense to split them up into PPN orders any more.

4.1 Killing off the Source

Killing off the source on the right hand side of the Poisson equation is the method used by chameleon and symmetron theories. This is achieved by adding a scalar potential to the action so that it is now

$$S = \int \mathrm{d}^4x\sqrt{-g}\frac{1}{8\pi G}\left[\frac{R}{2} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right] + S_{\mathrm{m}}[A^2(\phi)g_{\mu\nu}]. \tag{4.6}$$

In this case, the Einstein equations are still the same and we can still ignore the scalar's contribution to the energy-momentum tensor but the scalar's equation is modified to

$$\Box\phi = -8\pi GT\frac{\mathrm{d}\ln A}{\mathrm{d}\phi} + V'(\phi) = V'(\phi) + 8\pi G\frac{\mathrm{d}\ln A}{\mathrm{d}\phi}\rho. \tag{4.7}$$

The right hand side looks like an effective potential for the scalar:

$$V_{\mathrm{eff}} = V(\phi) + \rho\ln A \tag{4.8}$$

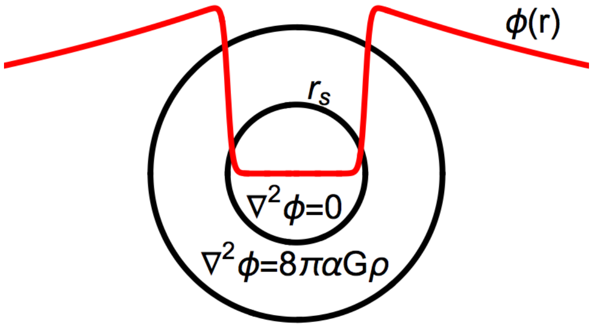

Figure 2. A spherical gravitational source. The entire object sources the gravitational fields χ_1 and Ψ but only the mass in the shell sources the scalar ϕ.

and we can use this to our advantage. The idea is the following. Suppose we choose $V(\phi)$ and $A(\phi)$ such that $V_{\rm eff}(\phi)$ has a minimum at some $\phi_{\rm min}$. $\phi_{\rm min}$ depends on ρ and so the minimum inside objects will be different from the minimum in the cosmological background. This means that inside the object, the field will want to move to reach the value $\phi_{\rm min}(\rho)$. Since $\phi_{\rm min}(\rho)$ is the solution of $V_{\rm eff}^\prime(\phi_{\rm min})=0$, the field equation is simply

$$\nabla^2\phi=0. \tag{4.9}$$

The equation for the field is unsourced and there is no scalar gradient and hence fifth-force. At some point, the field will have to move away from the minimum and towards its cosmological value and we expect some intermediate scale where we have fifth-forces. The essence of the screening mechanism is that it is possible to find parameters where this only happens in a very narrow shell near the surface. In this case, the field outside is only sourced by this very thin shell and the fifth-force is very heavily suppressed. This is shown schematically in figure 2.

Let's see how this works in practice. We start by defining a parameter α similar to the one above via

$$\alpha\equiv\left.\frac{{\rm d}\ln A}{{\rm d}\phi}\right|_{\phi=\phi_0} \tag{4.10}$$

and restrict to the case of spherical symmetry. We split the object into two regions separated by some screening radius $r_{\rm s}$ shown in figure 2. When $r<r_{\rm s}$ the field minimises its potential so that $\phi=\phi_{\rm min}$. There is no source for the field in this region and $\phi^\prime=0$. Outside, equation (4.7) becomes

$$\nabla^2\phi=V^\prime(\phi)+8\pi\alpha G\rho \quad r>r_{\rm s} \tag{4.11}$$

Let's make the further assumption that $V^\prime(\phi)$ can be neglected with $r>r_{\rm s}$. This is a valid assumption because $V^\prime(\phi)\approx m_0^2\phi$, where $m_0^2=V^{\prime\prime}(\phi_0)$ is the mass of the field in the cosmological background. This is typically of order H_0^{-2} whereas $\nabla^2\phi\sim\phi/R^2$, where R is the radius of the object and so the mass term is negligible compared with the Laplacian. In this case, the equation of motion reduces to

$$\frac{1}{r^2}\frac{\rm d}{{\rm d}r}\left(r^2\frac{{\rm d}\phi}{{\rm d}r}\right)=8\pi\alpha G\rho \quad r>r_{\rm s}. \tag{4.12}$$

This can easily be integrated using the fact that the mass enclosed inside a radius r is

$$M(r)=4\pi\int r^2\rho(r)\,{\rm d}r \tag{4.13}$$

Object	$\Phi_{\rm N}$
Earth	10^{-9}
The Sun	2×10^{-6}
Main-sequence stars	10^{-6}–10^{-5}
Local group	10^{-4}
Milky Way	$\mathcal{O}(10^{-6})$
Spiral and elliptical galaxies	10^{-6}–10^{-5}
Post-main-sequence stars	10^{-7}–10^{-8}
Dwarf galaxies	$\mathcal{O}(10^{-8})$

Table 1. The Newtonian potential of different astrophysical objects.

to find

$$F_5=\alpha\frac{\mathrm{d}\phi}{\mathrm{d}r}=2\alpha^2\frac{GM}{r^2}\left[1-\frac{M(r_{\rm s})}{M(r)}\right]\quad r>r_{\rm s}. \tag{4.14}$$

The familiar factor of $2\alpha^2$ is still there but now it is multiplied by a factor of

$$Q\equiv\left[1-\frac{M(r_{\rm s})}{M(r)}\right]. \tag{4.15}$$

When the screening radius is close to the radius of the object, R ,we have $M(r_{\rm s})\approx M$, where $M\equiv M(R)$ is the total mass of the object and so $Q\ll1$. In this case the object is screened. In the opposite limit where $r_{\rm s}=0$ we have $Q=1$ and the fifth-force is a factor of $2\alpha^2$ larger than the Newtonian one. In this case the object is unscreened. We will not show it here but the the screening radius is determined by the *self-screening parameter*[9]

$$\chi_0\equiv\frac{\phi_0}{2\alpha}. \tag{4.16}$$

In particular the screening radius is determined implicitly through the relation

$$\chi_0=4\pi G\int_{r_{\rm s}}^R r\rho(r)\,\mathrm{d}r. \tag{4.17}$$

When dealing with objects of mass M and radius R a good rule of thumb to determine whether they are screened or not is:

- If $\chi_0<GM/R$ the object is self-screening
- If $\chi_0>GM/R$ the object is at least partially unscreened.

This is a good criterion to use when deciding if a certain astrophysical system will be unscreened or not and gives us a good idea of where to test these theories. Note that for the Sun and the Milky Way $GM/R\sim10^{-6}$ and so $\chi_0<10^{-6}$ is a rough constraint found by requiring that they are screened. Dwarf galaxies have $GM/R\sim10^{-8}$ and so a lot of effort has been focused on looking at these systems as potential probes. The Newtonian potentials of some useful astrophysical objects are given in table 4.1.

[9]Technically, objects can be screened by their neighbours but we will not deal with this complication here.

Now let's go back and look at the solution outside the object, which is

$$\chi_1 = \Psi = \frac{GM}{r} \tag{4.18}$$

$$\alpha\phi = -\frac{Q(R)GM}{r}, \tag{4.19}$$

where $Q(R) = 1 - M(r_s)/M$. This gives the Jordan frame metric to 1PN as

$$\tilde{g}_{00} = -1 + 2U(1+Q) \tag{4.20}$$

$$\tilde{g}_{0i} = 0 \tag{4.21}$$

$$\tilde{g}_{ij} = [1 + 2U(1-Q)]\,\delta_{ij}. \tag{4.22}$$

Setting $G_N = (1+Q)G$ we can bring this into PPN form with

$$\gamma_{\text{PPN}} = \frac{1-Q}{1+Q}. \tag{4.23}$$

The difference now is that α can in principle be large because $Q \ll 1$ (provided $\chi_0 < 10^{-6}$). This means that this theory has no trouble passing the Cassini bound and it is still possible to have interesting effects on cosmological scales.

4.1.1 Two Examples: The Chameleon Mechanism

The first example of a theory that utilised this mechanism was chameleon screening. The coupling function and scalar potential are

$$V(\phi) = \frac{M^{4+n}}{\phi^n}, \quad A(\phi) = e^{\alpha\phi}, \tag{4.24}$$

with α constant. The effective potential is then

$$V_{\text{eff}}(\phi) = \frac{M^2}{\phi^n} + 8\pi\alpha G\rho, \tag{4.25}$$

which has a density-dependent minimum at

$$\phi(\rho) = \left(\frac{nM^2}{8\pi\alpha G\rho}\right)^{\frac{1}{n+1}}. \tag{4.26}$$

This is shown in figure 3. One can see that this potential has all of the properties that we need: If we consider a spherical over-dense object then the effective potential will have two minima, one inside the object and one inside the low-density background. Furthermore, the effective mass at the minimum is

$$m_{\text{eff}}^2 = V_{\text{eff}}''(\phi_{\text{min}}) = n(n+1)M^2\left(\frac{8\pi\alpha G\rho}{nM}\right)^{\frac{n+2}{n+1}}. \tag{4.27}$$

This is an increasing function of density and so the mass at the high density minimum can be several orders-of-magnitude larger than the mass at the low density minimum. Recall that the force-law for a massive scalar is of the Yukawa form

$$F \propto \frac{e^{-m_{\text{eff}}r}}{r^2}. \tag{4.28}$$

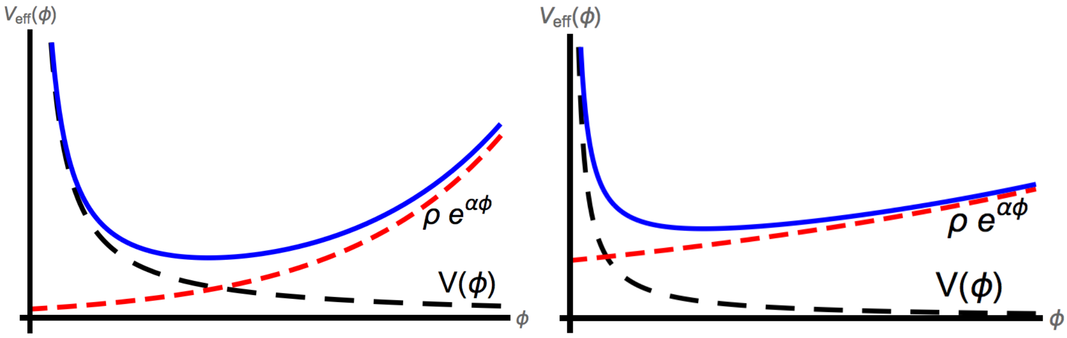

Figure 3. The effective potential for chameleon screening. The left panel shows the effective potential in low-desnity environments and the right shows the same potential in high-density environments. The Black dashed line shows the scalar potential $V(\phi)$ and the red dotted line shows the coupling function $\rho e^{\alpha\phi}$. The effective potential, which is the sum of the two, corresponds to the blue solid line.

In practice, the parameter M is chosen such that $m_{\rm eff} \lesssim \mathcal{O}(\mu\ \rm m)$ so that the fifth-force operates over ranges shorter than the most precise table-top experiments can probe. The non-linear field equation for the scalar opens up exciting possibilities of searching for chameleons using laboratory experiments with different geometries and densities but this is beyond the scope of these notes.

Before moving on, let's look at a very popular class of models: $F(R)$ theories. It turns out that these are chameleons in disguise. The action for these theories is

$$S = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\frac{f(R)}{16\pi G} + S_{\rm m}[\tilde{g}_{\mu\nu}], \tag{4.29}$$

where $R = R(\tilde{g})$. One can write this in an equivalent way using a new variable ψ:

$$S = \int \mathrm{d}^4x\frac{\sqrt{-\tilde{g}}}{16\pi G}\left[f(\psi) + f'(\psi)(R-\psi)\right] + S_{\rm m}[\tilde{g}_{\mu\nu}]. \tag{4.30}$$

If $f''(\psi) \neq 0$ the equation of motion for ψ is $\psi = R$, which can be put back into the action to recover (4.29). Next, we set $\Phi = f'(\psi)$ to find

$$S = \int \mathrm{d}^4x\frac{\sqrt{-\tilde{g}}}{16\pi G}\left[\Phi R - V(\phi)\right] + S_{\rm m}[\tilde{g}_{\mu\nu}], \tag{4.31}$$

where

$$V(\Phi) = \psi(\Phi)\Phi - f(\psi(\Phi)). \tag{4.32}$$

Finally, setting $\Phi = e^{-\sqrt{\frac{2}{3}}\phi}$ and applying a Weyl rescaling to $\tilde{g}_{\mu\nu}$ such that $\tilde{g}_{\mu\nu} = A^2(\phi)g_{\mu\nu}$ with $A(\phi) = e^{\frac{\phi}{\sqrt{6}}}$ one finds (after removing a total derivative proportional to $\Box\phi$)

$$S = \int \mathrm{d}^4x\frac{\sqrt{-g}}{8\pi G}\left[\frac{R(g)}{2} - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi)\right] + S_{\rm m}[A^2(\phi)g_{\mu\nu}], \tag{4.33}$$

with

$$V(\phi) = \frac{Rf'(R)-f(R)}{2f'(R)^2}. \tag{4.34}$$

Provided one chooses $f(R)$ such that $V(\phi)$ is of the chameleon form then the theory is a chameleon with $\alpha = 1/\sqrt{6}$. $f(R)$ theories are very popular in the literature because the coupling α is fixed and so there is only one free parameter: χ_0. In the language of $f(R)$ theories, people often work with the parameter $f_{R0} = f'(R)$ evaluated at the present time in the cosmological background. In terms of χ_0, one has $\chi_0 = 3/2 f_{R0}$. The most common example of an $f(R)$ theory that exhibits chameleon screening is the Hu & Sawicki model:

$$f(R) = R - m^2 \frac{c_1 (R/m^2)^n}{1 + c_2 (R/m^2)^n}. \tag{4.35}$$

Exercise:

Consider the Weyl rescaling:

$$\tilde{g}_{\mu\nu} = A^2(\phi) g_{\mu\nu}. \tag{4.36}$$

Show that:

$$\sqrt{-\tilde{g}} = A^4(\phi)\sqrt{-g} \tag{4.37}$$

$$R(\tilde{g}) = A^{-2}\left[R(g) - 6\Box\omega - 6g^{\mu\nu}\nabla_\mu\omega\nabla_\nu\omega\right], \tag{4.38}$$

with $\omega = \ln A$. Use this to transform equation (4.31) into equation (4.33). You will find Wald, appendix D useful.

4.1.2 Two Examples: The Symmetron Effect

The symmetron is described by the potential and coupling function

$$V(\phi) = -\frac{\mu^2\phi^2}{2} + \lambda\frac{\phi^4}{8\pi G}, \quad A(\phi) = 1 + \frac{\alpha\phi^2}{2} \tag{4.39}$$

so that the effective potential is of the $\mathbb{Z}_2$-symmetry breaking form

$$V_{\rm eff}(\phi) = \frac{\mu^2}{2}\left(\frac{8\pi\alpha G\rho}{\mu^2} - 1\right)\phi^2 + \lambda\frac{\phi^4}{8\pi G} = \frac{\mu^2}{2}\left(\frac{\rho}{\rho_\star} - 1\right)\phi^2 + \lambda\frac{\phi^4}{8\pi G}, \tag{4.40}$$

with $\rho_\star = \mu^2/8\pi\alpha G$. This is plotted in figure 4 for both $\rho > \rho_\star$ and $\rho < \rho_\star$. One can see that when $\rho < \rho_\star$ the $\mathbb{Z}_2$ symmetry is broken and the potential has a minimum at $\phi_\pm \approx \pm\mu\sqrt{2\pi G/\lambda}$. When $\rho > \rho_\star$ the symmetry is restored and the only minimum lies at $\phi = 0$. We want to screen inside high-density objects and so one typically chooses the model parameters such that $\rho_\star$ lies between the cosmological density $\rho_c \sim 3H_0^2 {M_{\rm pl}}^2\Omega_{\rm m0}$ and the central density of the object in question. When this is the case, the theory has all of the features we want: there are two distinct density-dependent minima. That being said, the screening mechanism is very different from the chameleon mechanism we saw above. In order to screen at sub-cosmological densities we require $\mu^2 \lesssim H_0^2$. The effective mass of the field is then $m_{\rm eff} \sim \mu \lesssim H_0$ and so the field is always very light. Instead of altering the range of the force, the symmetron screens the force because the coupling $\alpha(\phi) \approx 0$. The fifth-force (4.5) is

$$\vec{F}_5 = -\alpha(\phi)\nabla\phi \approx \alpha\phi\nabla\phi. \tag{4.41}$$

Provided the field has reached its symmetry restoring minimum i.e. $\phi = 0$ this is identically zero.

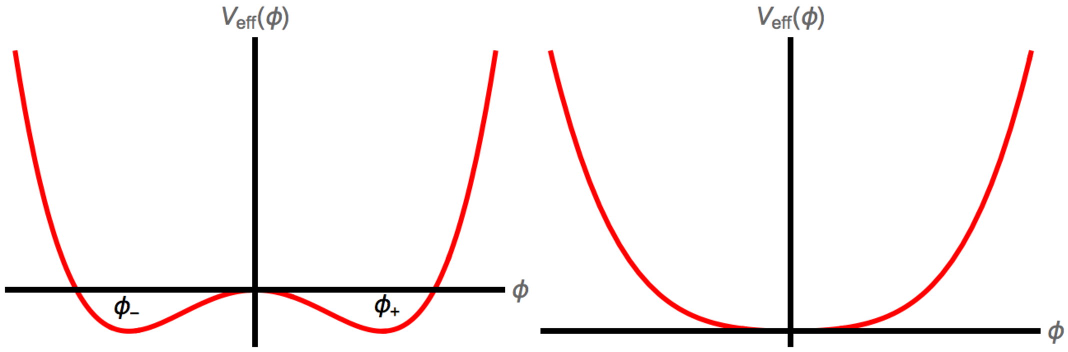

Figure 4. The effective potential for the symmetron. The left panel shows the low-density, symmetry broken phase and the right panel shows the high-density, symmetry restoring phase.

4.2 New Derivative-Interactions: The Vainshtein Mechanism

Another way to suppress scalar forces is to alter the Poisson equation to include additional derivative interactions. For example, consider the Poisson equation for a spherically symmetric body

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = 8\pi\alpha G r^2\rho. \tag{4.42}$$

This is a total derivative that can be integrated to give,

$$r^2\phi = \alpha GM \Rightarrow \alpha\nabla\phi = 2\alpha^2\frac{GM}{r^2} \tag{4.43}$$

which is how we recover the inverse-square law. One can see that $F_5/F_{\rm N} = 2\alpha^2$ as we found above. Suppose then that we were to have a different operator other than the Laplacian, for example

$$\frac{1}{\Lambda^4}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^3\right] = 8\pi\alpha G r^2\rho, \tag{4.44}$$

where we have included a new mass-scale Λ for dimensional consistency. This too is a total derivative and can be integrated to give

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = \left(2\alpha\Lambda^4 GM\right)^{\frac{1}{3}}. \tag{4.45}$$

One can see that the scalar force-law is very different from inverse-square, in fact, it is constant. The ratio of the fifth- to Newtonian force (GM/r^2) is then

$$\frac{F_5}{F_{\rm N}} = \frac{\alpha\,\mathrm{d}\phi/\,\mathrm{d}r}{\mathrm{d}\Phi_{\rm N}/\,\mathrm{d}r} = 2\alpha^2\left(\frac{r}{r_{\rm V}}\right)^2 \quad r_{\rm V}^3 \equiv 2\alpha\frac{GM}{\Lambda^2}. \tag{4.46}$$

$r_{\rm V}$ is known as the *Vainshtein radius* and it can be made as big as we like because Λ is a free mass scale that we can make as small as we like. The fifth-force is then highly suppressed whenever $r < r_{\rm V}$. This is the Vainshtein mechanism. We will give an example of a common theory that exhibits it below.

4.2.1 Example: Cubic Galileons

Galileon theories have received a lot of theoretical attention lately due to their links with massive gravity and powerful non-renormalisation theorems. Here, we are only interested in their small-scale behaviour. Galileon theories are invariant under the galileon symmetry $\phi \rightarrow \phi + b_\mu x^\mu + c$ when expanded around Minkowski space and it turns out there are four possible derivative operators that one can write down in four dimensions. The simplest is the cubic galileon:

$$S = \int \mathrm{d}^4x \frac{\sqrt{-g}}{8\pi G}\left[\frac{R}{2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{\Lambda^2}\partial_\mu\phi\partial^\mu\phi\Box\phi\right] + S_{\rm m}[\tilde{g}_{\mu\nu}], \quad \tilde{g}_{\mu\nu} = e^{2\alpha\phi}g_{\mu\nu}. \tag{4.47}$$

The name *cubic galileon* derives from the fact that the new operator action contains thee fields. Similarly, there are quartic and quintic galileons. Despite being higher-order, the galileon shift symmetry ensures that the equations of motion are second-order and the theory is therefore free of the Ostrogradski ghost instability. Since we are working in the Einstein frame, the equations of motion for the metric are Einstein's equations and the scalar equation of motion is

$$\Box\phi + \frac{2}{\Lambda^2}\left[(\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right] = 8\pi\alpha G\rho. \tag{4.48}$$

In the case of a static spherically symmetric configuration this becomes

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{r}{\Lambda^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^2\right] = 8\pi\alpha G\rho. \tag{4.49}$$

We can recognise the first term as the usual contribution from the Laplacian. The second term is the new contribution from the cubic galileon operator. This can be integrated once to give

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{1}{\Lambda^2 r}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)^2 = 2\alpha\frac{GM}{r^2}. \tag{4.50}$$

Note that the right hand side is 2α times the Newtonian force and so we can begin to see the Vainshtein mechanism at work. In the absence of the cubic galileon we are back to the case studied above where $F_5 = 2\alpha^2 F_{\rm N}$, but the presence of the galileon term changes this. One can already see that the second term dominates at small distances but let's put this on a more concrete footing. Introducing the Vainshtein radius

$$r_{\rm V}^3 = \frac{GM}{\alpha\Lambda^2} \tag{4.51}$$

and setting $\mathrm{d}\phi/\,\mathrm{d}r = F_5/\alpha$ we can divide by $F_{\rm N} = GM/r^2$ to find

$$\frac{F_5}{F_{\rm N}} + \left(\frac{r_{\rm V}}{r}\right)^3\left(\frac{F_5}{F_{\rm N}}\right)^2 = 2\alpha^2. \tag{4.52}$$

One can then see the Vainshtein mechanism at work. When $r \ll r_{\rm V}$ the new term dominates over the Laplacian and we have

$$\frac{F_5}{F_{\rm N}} = 2\alpha^2\left(\frac{r}{r_{\rm V}}\right)^{\frac{3}{2}} \tag{4.53}$$

whereas when $r \gg r_{\rm V}$ the new term is suppressed and we recover the inverse-square behaviour. This is shown in figure figure 5.

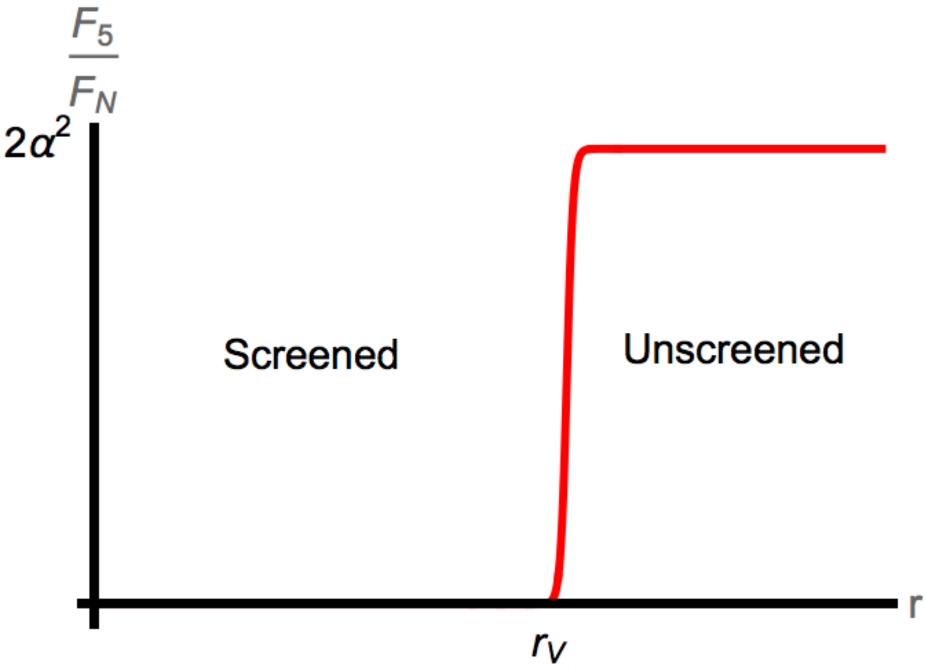

Figure 5. Vainshtein screening of a spherical object.

Exercise:

Use equation (4.48) to derive equation (4.49). The Christoffel symbols are not zero, despite the fact that we are working in a flat space-time and you will need to use this.

Whether or not an object screens then depends on the Vainshtein radius. Lunar Laser Ranging measures the motion of the Moon about the Earth very precisely and constrains deviations in the gravitational potential:

$$\frac{\delta\Phi_{\rm N}}{\Phi_{\rm N}} < 2.4 \times 10^{11}. \tag{4.54}$$

Using equation (4.53) we have

$$\frac{\delta\Phi_{\rm N}}{\Phi_{\rm N}} = 2\alpha^2\left(\frac{r}{r_{\rm V}}\right)^{\frac{3}{2}}. \tag{4.55}$$

The Earth-Moon distance is 3.84×10^8 m and so one finds $r_{\rm V}^{\oplus} > 7.3 \times 10^{15}$m $\sim \mathcal{O}(10^{-1})$ Pc for $\alpha = 1$. Now note that

$$r_{\rm V}^3 = \frac{r_{\rm S}}{\Lambda^2}, \tag{4.56}$$

where $r_{\rm S}$ is the Schwarzchild radius. This is 9 mm for the earth and so one has $\Lambda^2 = 2.3 \times 10^{-53}$ mm^{-2} if we assume the LLR bound is just satisfied. The Schrwazchild radius of the Sun is ~ 3 km and so we find $r_{\rm V}^{\odot} \sim \mathcal{O}(\rm pc)$. The Vainshtein radius of the Sun is larger than the solar system. One can see that Vainshtein screening is incredibly efficient. This is both a blessing and a curse: deviations from general relativity are well hidden but finding novel probes of the mechanism is incredibly difficult.

4.3 Equivalence Principle Violations

We end this chapter by discussing the most important difference between the two screening mechanisms: violations of the equivalence principle. General relativity was founded on the equivalence principle: The motion of test-bodies is independent of their structure and composition. This appears in the Newtonian limit of general relativity as follows. We know that the geodesic equation gives us the force-law

$$M\ddot{\vec{x}} = -M\nabla\Phi_{\rm N}^{\rm ext} \tag{4.57}$$

in the non-relativistic limit. Here, $\Phi_{\rm N}^{\rm ext}$ refers to the Newtonian potential generated by some external body e.g. the Sun. The mass on the left-hand side is the *inertial mass*, it is the mass we put into the point-particle action

$$S = -M\int {\rm d}s\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}, \tag{4.58}$$

where in this one instance a dot denotes a derivative with respect to the proper time s. The mass on the right hand side is the *gravitational mass*. This should be thought of more as a gravitational charge in analogy with electromagnetism. It tells us how strongly a test-body responds to an external gravitational field $\Phi_{\rm N}^{\rm ext}$. The fact that $M_{\rm LHS} = M_{\rm RHS}$ means that they cancel and the motion of a non-relativistic particle is independent of its mass. This is the equivalence principle. In scalar-tensor theories, (4.57) is generalised to

$$M\ddot{\vec{x}} = -M\nabla\Phi_{\rm N}^{\rm ext} - Q\nabla\phi^{\rm ext}, \tag{4.59}$$

where Q is the scalar gravitational charge. One can see that if $Q = M$ or $Q = 0$ the equivalence principle is satisfied. Deriving the value of Q in these theories is a long and cumbersome process and so rather than provide a hand-wavy, incorrect proof, I will simply state the results here:

$$\begin{aligned} Q &= M\left[1-\frac{M(r_{\rm s})}{M}\right], && \text{Chameleons} \\ Q &= M && \text{Vainshtein.} \end{aligned} \tag{4.60}$$

Chameleon-like theories do not satisfy the equivalence principle because $Q \neq M$ unless the screening radius is zero or equal to the radius of the object i.e. unless the object is fully screened or fully unscreened. Theories that screen using the Vainshtein mechanism do satisfy the equivalence principle[10]. The equivalence principle is very well tested in the solar system and this allows one to constrain chameleon theories but not those that screen using the Vainshtein mechanism.

5 Non-Relativistic Stars: A Laboratory for Testing Fundamental Physics

In this final chapter, we will look at the structure of non-relativistic stars and see how they can be used to test alternate theories of gravity.

[10] The one exception to this is black holes but we are not interested in highly relativistic objects here.

5.1 Stellar Structure Equations

We begin by deriving the equations of stellar structure. Non-relativistic stars are spherical to a good approximation and, we can treat them as a perfect fluid described by their Eulerian pressure P and density ρ. Working in the rest-frame of the star, we have $\gamma = 1$ and so $u^\mu = (1, \vec{0})$. The energy momentum tensor (3.11) is then

$$T^{\mu\nu} = \mathrm{diag}(\rho, P, P, P). \tag{5.1}$$

The space-time is

$$\mathrm{d}s^2 = -(1-2\Phi)\,\mathrm{d}t^2 + (1+2\Psi)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{5.2}$$

The equation of motion for the fluid is given by the zero-component of

$$\nabla_\mu T^{\mu\nu} = 0, \tag{5.3}$$

which results in

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}. \tag{5.4}$$

Note that we have not specified the theory of gravity, all we have done is assume that $T^{\mu\nu}$ is conserved, which means we are working in the Jordan frame. The theory of gravity determines $\mathrm{d}\Phi/\,\mathrm{d}r$ so let's specialise to general relativity for now and set $\mathrm{d}\Phi/\,\mathrm{d}r = GM(r)/r^2$. This gives us the hydrostatic equilibrium equation:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM(r)\rho(r)}{r^2}. \tag{5.5}$$

We also have an equation for $M(r)$:

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi G r^2 \rho(r) \tag{5.6}$$

but this does not close the system of equations and allow us to solve for $P(r)$ and $\rho(r)$. In order to do this we must specify non-gravitational physics. First, we know that in order to support themselves against gravitational collapse, stars must burn fuel in their centres. If ϵ_i is the energy released per unit mass from the i^{th} burning process in the core and $\bar{\epsilon}_i$ is the energy lost in that process (e.g. from neutrinos produced in the reactions, that stream away) then the luminosity, the energy released per unit time, is given by

$$\frac{\mathrm{d}L(r)}{\mathrm{d}t} = 4\pi r^2 \rho(r)\left(\sum_i \epsilon_i - \sum_i \bar{\epsilon}_i\right). \tag{5.7}$$

This is the energy generation equation. The surface luminosity $L \equiv L(R)$ is one of the most important stellar parameters since it is directly observable. Another observable quantity is the effective temperature T_{eff}. The temperature gradient in a star is given by the radiative transport equation

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{4a}\frac{\kappa(\rho,T)}{T^3}\frac{\rho L}{4\pi r^2}. \tag{5.8}$$

Here a is a constant that appears in the pressure law for radiation (we will see this explicitly in a moment) and $\kappa(\rho, T)$ is the opacity (the cross-section for radiation absorption per unit mass). These equations still do not close as one needs to supply equations of state of the form $\kappa(\rho, T)$, $P(\rho, T)$ and $\epsilon_i = \epsilon_i(\rho, T)$. In practice, complicated numerical codes are needed to solve these problems but there are several simplifying assumptions one can make in order to gain valuable physical insights. The pressure in most stars is primarily due to the internal motions of the gas particles as described by the idea gas law:

$$P_{\rm gas} = \frac{\rho k_{\rm B} T}{\mu m_{\rm H}}, \tag{5.9}$$

where μ is the mean molecular mass and $m_{\rm H}$ is the mass of Hydrogen, and the pressure due to absorbing radiation in the interior:

$$P_{\rm rad} = \frac{1}{3} a T^4. \tag{5.10}$$

In what follows we will consider these pressure laws only. Furthermore, to avoid dealing with atmospheric models we will define the radius of the star R as the radius where the pressure falls to zero i.e. $P(r) = 0$.

5.2 Scale-Invariance of the Stellar Structure Equations

One can make a lot of progress by noting that the stellar structure equations are scale-invariant. To see this, consider the hydrostatic equilibrium equation (5.5). Since P has units of GM/r^4 we can see that if one scales $P \to P_{\rm c} x_P$, where $P_{\rm c}$ is the central pressure, $r \to R x_r$, $M(r) \to M x_M$ and $\rho(r) \to x_\rho M/R^3$, where x_i are dimensionless functions we have

$$P_{\rm c} \frac{{\rm d}x_P}{{\rm d}x_r} = -\frac{GM^2}{R^4} \frac{x_\rho x_M}{x_r^2}. \tag{5.11}$$

Now since x_i are dimensionless we can immediately see that

$$P_{\rm c} \propto \frac{GM^2}{R^4}. \tag{5.12}$$

We have a relation for how $P_{\rm c}$ scales with G, M and R without having to have solved any equations. Let's see how far we can push this. Doing the same thing for the radiative transfer equation we find

$$L \propto \frac{R^4 T^4}{M}. \tag{5.13}$$

Now we need to decide what our equation of state is. If we assume that the star is gas dominated we find

$$P \propto \frac{MT}{R^3} \Rightarrow T = \frac{GM}{R} \tag{5.14}$$

using equation (5.9) and (5.12). If we instead assume it is radiation dominated we have, from equation (5.10) and (5.12)

$$P \propto T^4. \tag{5.15}$$

Inserting these into equation (5.13) we have

$$\begin{aligned} &L \propto G^4 M^3 \quad &\text{gas} \\ &L \propto GM \quad &\text{radiation.} \end{aligned} \tag{5.16}$$

This gives us a mass-luminosity relation and shows us how the luminosity scales with the strength of gravity. One can see that gas-supported stars are more sensitive to changes in G

than radiation-supported stars. Furthermore, theories of gravity that predict stronger gravity than general relativity predict that stars are more luminous whilst the converse is true for theories that predict weaker gravity. Physically, this is because stronger gravity means fuel must be burnt at a higher rate to provide the extra pressure gradient needed to stave off gravitational collapse and hence the rate of energy release increases. Using equation (5.9) and (5.10) one finds

$$\frac{P_{\rm rad}}{P_{\rm gas}} \propto \frac{T^3}{\rho} \propto G^3 M^2 \tag{5.17}$$

and so one can see that low mass stars are gas-supported and high mass stars are radiation-supported. Low mass stars are therefore a better probe of modified gravity[11]. One can also see that increasing G makes stars more radiation supported at fixed mass.

5.3 Ploytropic Equations of State: The Lane-Emden Equation

One of the simplest choices for the equation of state is

$$P = K\rho^{\frac{n+1}{n}}. \tag{5.18}$$

This is known as a *polytropic* equation of state and n is known as the *polytropic index*. Polytropes are very important in stellar physics and can describe a variety of different stars. For example, $n = 3$ describes low mass main-sequence stars and fully relativistic (white dwarf) stars, $n = 1.5$ describes fully convective stars and $n = 5$ can be used to model globular clusters. Let's see what happens to the stellar structure equations if we assume this equation of state. First, we define a few important quantities. Since the stellar structure equations are scale invariant[12] we can reduce the system to dimensionless form. First, we define the dimensionless radial coordinate ξ via:

$$r = r_0\xi, \quad r_0^2 = \frac{(n+1)P_{\rm c}}{4\pi G\rho_{\rm c}^2}, \tag{5.19}$$

where a c refers to central quantities. Since the stellar radius is defined as the point where $P(R) = 0$ it is useful to define ξ_R via $\theta(\xi_R) = 0$. The stellar radius is then $R = r_0\xi_R$. We also define the dimensionless function $\theta(\xi)$ via

$$\begin{aligned} P &= P_{\rm c}\theta^{n+1} \\ \rho &= \rho_{\rm c}\theta^n. \end{aligned} \tag{5.20}$$

This gives us the relation $P_{\rm c} = K\rho_{\rm c}^{\frac{n+1}{n}}$. Dividing equation (5.5) by $G\rho/r^2$ we have

$$\frac{r^2}{G\rho}\frac{{\rm d}P}{{\rm d}r} = -M(r), \tag{5.21}$$

which can be differentiated once using equation (5.6) to find

$$\frac{1}{r}\frac{\rm d}{{\rm d}r}\left(\frac{r^2}{4\pi G\rho}\frac{{\rm d}P}{{\rm d}r}\right) = -\rho(r). \tag{5.22}$$

[11] Provided of course that one is trying to test it using the stellar luminosity.

[12] Actually, the symmetry is even larger than this. The stellar structure equations with a polytropic equation of state are *homology*-invariant. This means that once you know one solution with one boundary condition you actually know other solutions with different boundary conditions. This goes beyond the scope of the course but it allows one to analyse the equations using very powerful mathematical techniques.

Inserting the equation of state (5.18) and changing to dimensionless variables using (5.19) and (5.20) we arrive at the famous Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n. \tag{5.23}$$

This equation is non-linear and can only be solved in a few special cases, all of which are uninteresting for testing gravity. These are given in box 1 for completeness. In practice one needs to solve this numerically but this is simple and can be done using elementary techniques such as the Runge-Kutta family of methods. We still need to specify the boundary conditions. We know that the central pressure is given by P_{c} and so we clearly have $\theta(0) = 1$. The second boundary condition is forced on us by spherical symmetry: the pressure must go to zero smoothly at $r = 0$ and so we have $\theta'(0) = 0$. Some examples of the solutions satisfying these conditions are shown in figure 6. With these conditions, one can show that the behaviour of θ near the origin is

$$\theta(\xi) = 1 - \frac{\xi^2}{6} + \frac{n}{120}\xi^4. \tag{5.24}$$

This is useful when integrating the equations numerically because one cannot typically specify the boundary condition at $\xi = 0$ but instead must integrate from $\xi = \delta$ where $\delta \ll 1$.

Box 1: Exact solutions of the Lane-Emden equation

$$\begin{aligned} n = 0: &\quad \theta(\xi) = C_0 - \tfrac{C_1}{\xi} - \tfrac{1}{6}\xi^2 \\ n = 1: &\quad \theta(\xi) = C_0\tfrac{\sin\xi}{\xi} + C_1\tfrac{\cos\xi}{\xi} \\ n = 5: &\quad \theta(\xi) = \tfrac{1}{\sqrt{1+\xi^2/3}}, \end{aligned} \tag{5.25}$$

where C_i are integration constants.

5.3.1 The Mass Radius Relation and the Chandrasekhar Mass

Typically, we cannot see inside stars and so, in some sense, P_{c} and ρ_{c} are meaningless quantities. What we observe are quantities such as the mass, radius and luminosity. Polytropic equations of state predict a mass-radius relation that we will derive here. First, we can integrate equation (5.6) to give

$$\begin{aligned} M &= \int_0^R 4\pi r^2\rho(r)\,\mathrm{d}\xi = 4\pi r_0^3\rho_{\mathrm{c}}\int_0^{\xi_R}\xi^2\theta^n(\xi)\,\mathrm{d}\xi = -4\pi r_0^3\rho_{\mathrm{c}}\int_0^{\xi_R}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) \\ &= 4\pi\omega_R r_0^3\rho_{\mathrm{c}}, \end{aligned} \tag{5.26}$$

where we have used the Lane-Emden equation to replace θ^n and have defined

$$\omega_R \equiv -\xi_R^2\left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=\xi_R}, \tag{5.27}$$

which is a dimensionless number that must be computed numerically. Using (5.19) we find

$$M = 4\pi\xi_R^2\omega_R\left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}}\rho_{\mathrm{c}}^{\frac{n-3}{2n}}, \tag{5.28}$$

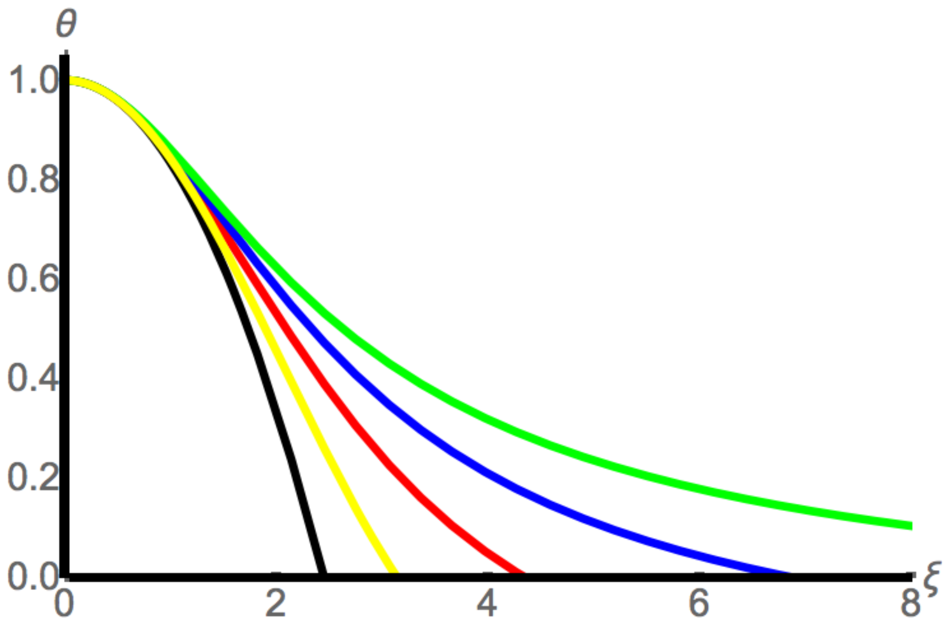

Figure 6. Solutions of the Lane-Emden equation when $n = 0$ (black), $n = 1$ (yellow), $n = 2$ (red), $n = 3$ (blue) and $n = 4$ (green).

which can be inverted to give

$$\rho_c = \left[\frac{M}{4\pi\omega_R} \left(\frac{4\pi G}{(n+1)K} \right)^{\frac{3}{2}} \right]^{\frac{2n}{3-n}}. \tag{5.29}$$

Next, recall that

$$R = r_0 \xi_R = \left[\frac{(n+1)K}{4\pi G} \right]^{\frac{1}{2}} \xi_R \rho_c^{\frac{1-n}{2n}}. \tag{5.30}$$

Substituting for the central density using (5.29) we find

$$R \propto M^{\frac{n-1}{n-3}}. \tag{5.31}$$

This is the mass-radius relation. Note that for $n = 1$ the radius is independent of the mass and for $n = 3$ i.e. a fully relativistic star, the mass is independent of the radius. This is the origin of the Chandrasekhar mass: an upper limit for the mass of a white dwarf star. Non-relativistic stars have $n = 1.5$ and fully relativistic stars have $n = 3$. When $n = 3$ and the star is fully relativistic one has

$$M = \frac{5.82}{\mu_e^2} M_\odot, \tag{5.32}$$

independent of the central density. Here μ_e is the number of free electrons per atom. To derive this we have used the numerical value $K = 1.24 \times 10^{15} \mu_e^{-4/3}$ (in cgs units), which comes from statistical physics applied to a fully relativistic electron gas. If the mass of the star is less than this it cannot be fully relativistic but, since any degenerate star cannot have $n > 3$, this represents an upper limit for the mass: stars with a larger mass are unstable and collapse to form neutron stars. A helium white dwarf has $\mu_e = 2$ because helium contributes two electrons per atom and this gives an upper limit

$$M_{\rm Ch} \approx 1.4 M_\odot. \tag{5.33}$$

5.4 Main-Sequence Stars: The Eddington Standard Model

Let's now turn to more familiar stars: stars like the Sun. These are low mass stars that burn hydrogen in their core. As we will see now, these are well described by Lane-Emden models provided that we make one simple assumption: the Eddington approximation. Recall from equation (5.17) that the ratio of the radiation to gas pressure in a star is proportional to T^3/ρ. This is actually a very important quantity: it is the specific entropy i.e. the entropy per unit mass. The Eddington approximation assumes that this is constant and hence the ratio of the gas to radiation pressure is a constant. This is a good approximation for low mass main-sequence stars that are not convective. We can then define the constant β via

$$\beta \equiv \frac{P_{\rm gas}}{P} \Rightarrow P_{\rm rad} = (1-\beta)P. \tag{5.34}$$

Equating $P_{\rm gas}/\beta$ with $P_{\rm rad}/(1-\beta)$ we find

$$\frac{T^3}{\rho} = 3a\frac{k_{\rm B}}{\mu m_{\rm H}}\frac{1-\beta}{\beta}. \tag{5.35}$$

The total pressure is then

$$P = \frac{\rho k_{\rm B} T}{\mu m_{\rm H}} + \frac{1}{2}aT^4 = K(\beta)\rho^{\frac{4}{3}}, \tag{5.36}$$

with

$$K(\beta) = \left(\frac{3}{a}\right)^{\frac{1}{3}}\left(\frac{k_{\rm B}}{\mu m_{\rm H}}\right)^{\frac{4}{3}}\left(\frac{1-\beta}{\beta^4}\right)^{\frac{1}{3}}. \tag{5.37}$$

Currently, β is an unknown and so we want to relate it to an observable quantity: the mass. Using equation (5.28) and the definition of r_0 (5.19) we find

$$\frac{1-\beta}{\beta^4} = \left(\frac{M}{M_{\rm edd}}\right)^2, \tag{5.38}$$

where the Eddington mass is

$$M_{\rm Edd} = \frac{4\omega_R}{\sqrt{\pi}G^{\frac{3}{2}}}\left(\frac{k_{\rm B}}{\mu m_{\rm H}}\right)^2\left(\frac{3}{a}\right)^{\frac{1}{2}} \approx 18.2\mu^{-2}. \tag{5.39}$$

Equation (5.38) is a quartic equation that can be solved numerically to find the value of β for a star of mass M given the mean molecular mass μ. This is 1/2 for fully ionised hydrogen since the contribution to the density from electrons is negligible.

Next, we want to compute something observable: the luminosity. Using $P = P_{\rm rad}/(1-\beta)$ in equation (5.5) and using equation (5.8) we find

$$L = \frac{4\pi(1-\beta)GM}{\kappa}, \tag{5.40}$$

where $L = L(R)$ is the surface luminosity. The opacity in hydrogen burning stars is due to electron scattering, for which κ is a constant that is independent of T and ρ. This means that specifying the mass of a star alone is enough to determine its luminosity provided we make some assumptions about its composition and opacity: given the mass, one can solve Eddington's quartic equation for β and hence find the luminosity using equation (5.40).

Finally, we note several drawbacks of the Lane-Emden approach:

- The Lane-Emden approach predicts that the surface temperature of the star is zero. Since T^3/ρ is constant we have $T \propto \theta$ for $n = 3$, which is zero at the surface of the star. In practice, the effective temperature (or $(B - V)$) is one of the most important observational properties of a star.

- Nuclear burning was not included. This means that the star is really just a ball of gas supporting itself against gravitational collapse through some pressure that we have specified by hand. For this reason, the star will not evolve in time and we can learn anything about stellar evolution. The lack of nuclear burning means the mass-radius and mass-luminosity relation we predicted is not quite correct.

- Many post-main-sequence stars are layered, both in terms of their composition and their energy transfer (radiative or convective). Polytropic models assume a homogeneous fluid but this can be accounted for by stitching together fluids with different values of n.

That being said, it is these drawbacks that make polytropic models perfect tools for studying alternate theories of gravity. The non-gravitational physics has been removed, which is perfect for disentangling the effects of modified gravity from thermodynamic processes.

5.5 Application to Alternate Theories of Gravity

So far we haven't actually applied this to alternate theories of gravity. As remarked earlier, one can do this by substituting the modified solutions for $\mathrm{d}\Phi/\,\mathrm{d}r$ into the hydrostatic equilibrium equation (5.4) and repeating the derivations above. This is a complicated task for the alternate theories we have discussed in this course; each one requires a separate paper. Here, we will do something simpler and assume that we are working with an alternate theory where

$$G = \lambda G_{\mathrm{N}}, \tag{5.41}$$

where λ is a dimensionless parameter that is given by the theory[13]. The strength of gravity entered into the previous section in two places: the definition of the Eddington mass and the formula for the total luminosity. Recall from equation (5.39) that $M_{\mathrm{Edd}} \propto G^{-3/2}$. This means that

$$M_{\mathrm{Edd}}(\lambda) = \lambda^{-\frac{3}{2}} M_{\mathrm{Edd}}^{\mathrm{GR}} = 18.2\lambda^{-\frac{3}{2}}\mu^{-2}. \tag{5.42}$$

One therefore has to solve the modified form of Eddington's quartic equation,

$$\frac{1-\beta}{\beta^4} = \lambda^3 \left(\frac{M}{M_{\mathrm{Edd}}}\right)^2, \tag{5.43}$$

[13] We learned in previous lectures that non-relativistic objects always respond to G_{N} after one transforms to the PPN gauge and so this shift in G is difficult to achieve in classical alternate theories of gravity. That being said, we have also seen that this can be realised in modern theories such as chameleons and galileons where stars in our solar system would respond to $G = G_{\mathrm{N}}$ but stars in other galaxies can feel an effective value of G different from G_{N} as measured in the solar system. This simple model is then a good proxy for theories with screening mechanisms.

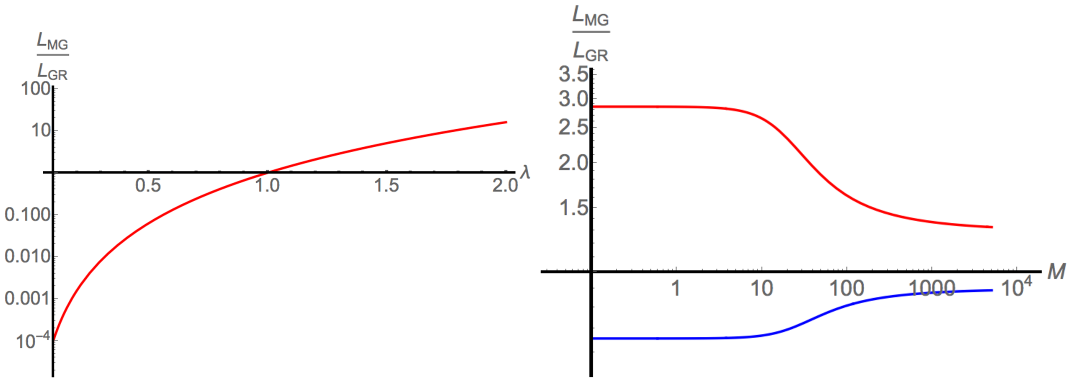

Figure 7. Left panel: The ratio of the luminosity of a solar mass star in modified gravity to the general relativity prediction as a function of λ. Right panel: The same ratio as a function of mass for $\lambda = 1.3$ (red) and $\lambda = 0.9$ (blue).

for $\beta(\lambda)$. Using equation (5.40), we then have

$$\frac{L_{\rm MG}}{L_{\rm GR}} = \frac{[1-\beta(\lambda)]\lambda}{[1-\beta(\lambda=1)]}. \tag{5.44}$$

This is plotted for a one solar mass star as a function of λ in figure 7. One can see that when $\lambda > 1$, i.e. gravity is stronger than general relativity predicts, the luminosity is enhanced whereas the converse behaviour is exhibited when $\lambda < 1$. This is in line with what we discussed above: stronger gravity requires a faster rate of nuclear burning to prevent collapse and hence a higher rate of energy generation; weaker gravity does not need to burn as quickly and hence energy is produced at a lower rate. The ratio is plotted for $\lambda = 1.3$ and $\lambda = 0.9$ in figure 7. One can see the same behaviour where $\lambda > 1$ enhances the luminosity and $\lambda < 1$ reduces it. One can also see that low mass stars are more affected by changing G than high mass stars. This is also what we predicted above where we found that the mass-luminosity relation for low-mass stars is $L \propto G^4$ but is only $L \propto G$ for high-mass stars. In fact, it is clear from the figure that the luminosity ratio tends to these values (i.e. λ^4 and λ) in the limiting cases $M \to 0$ and $M \to \infty$.

5.6 Stars as Probes of Fundamental Physics

Although this course has been primarily concerned with testing alternate theories of gravity we end by remarking that many other physical processes apart from gravity are involved in determining the structure and evolution of stars. Take, for example, the electron scattering opacity that appears in the formula for the luminosity of main-sequence stars (5.40). If we calculate this from first principles we have

$$\kappa_{\rm es} \propto \left(\frac{e^2}{m_{\rm H}c^2}\right)^2. \tag{5.45}$$

This means that any spatial or temporal variation in the speed of light, the mass of hydrogen or the electron charge will show up in the luminosity of main-sequence stars, either as a function of space or time.

As another example, consider the energy generation equation (5.7). In the standard model, only neutrinos contribute energy losses in the form of $\bar{\epsilon}$. Neutrinos free-stream out of the star and the energy they carry with them is lost. This determines the life-time of the main-sequence because the star needs to burn fuel at a slightly higher rate than it would have to if there were no losses to stop itself collapsing. Now suppose we had a new particle that is also produced and that also free streams out. This would be an extra source of energy loss and would reduce the lifetime of the Sun. Since we know the Sun is 4.6 billion years old we can't add new particles that would reduce the main-sequence life-time below this. One popular dark matter candidate is axions, which couple to photons. Photons deep in the Sun's core can convert into axions via the Primakoff process, which then free-stream out of the Sun causing energy to be lost. The age of the Sun then lets us constrain the coupling of the axion to photons.

Stars are finely balanced engines that utilise fundamental physics to convert hydrogen into helium and beyond. They truly are natures laboratories!